PLANCKS Test 2020

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Casimir effect

It has been experimentally verified that two uncharged parallel conducting plates placed very close to one another (at separation of order μm) experiences an attractive force *in vacuum*. This result is a *bona fide* prediction of quantum field theory (QFT), as it cannot be explained classically or by non-relativistic quantum mechanics.

Remark: in this question *no prior knowledge of QFT is assumed*[1]. **Throughout this question we will use units $c = \hbar = 1$, as a PLANCKS member should.**

(a) Consider a classical massless scalar field in (1+1)-dimensional spacetime $\phi(t, x)$, subject to Dirichlet boundary condition

$$\phi(t, x = 0) = \phi(t, x = L) = 0\,. \tag{1}$$

Show that if this field satisfies the wave equation

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}\right)\phi(t, x) = 0\,, \tag{2}$$

the general solution can be written in the form of

$$\phi(t, x) = \sum_{n=-\infty}^{\infty}\left(\alpha_n u_n(t, x) + \beta_n u_n^*(t, x)\right), \tag{3}$$

[1]This can be taken to be an algorithmic way to do simple QFT calculation. For more details, please consult a QFT textbook after the test, e.g. by Blundell/Peskin/Weinberg/Ryder/Srednicki.

where $u_n(t,x) = Ne^{-i\omega_n t}\sin\omega_n x$ and N is some normalization constant that does not depend on n, and show that for real scalar field, $\beta_n = \alpha_n^*$.

Solution: This is standard method to find the solution to wave equation using Fourier series. Since plane waves satisfy the wave equations, the general solutions are given by

$$\phi(t,x) = \sum_{n=-\infty}^{\infty}\left(\alpha_n e^{-i|\omega_n|t+i\omega_n x} + \beta_n e^{-i|\omega_n|t+i\omega_n x}\right) \tag{4}$$

Imposing the boundary condition, we obtain the solution with $\omega_n = n\pi/L$. Note that for real scalar field, $\beta_n = \alpha_n^*$ since for every n, each term will be of the form $2\mathrm{Re}(\alpha_n u_n(t,x))$.

(b) The **Klein-Gordon inner product** is given by

$$(f,g) \coloneqq -i\int_0^L \mathrm{d}x\left(f\frac{\partial g^*}{\partial t} - \frac{\partial f}{\partial t}g^*\right), \tag{5}$$

find the normalization constant N using the orthonormality of the basis functions $(u_m, u_n) = \delta_{m,n}$.

Solution: Using the inner product, we get

$$1 = 2N^2|\omega_n|\frac{L}{2}. \tag{6}$$

Therefore we get

$$N = \frac{1}{\sqrt{L\omega_n}} = \frac{1}{\sqrt{n\pi}}. \tag{7}$$

(c) We obtain a quantized scalar field by promoting α_n to an annihilation operator $\hat{a}_n$, and α_n^* to a creation operator $\hat{a}_n^\dagger$. In this sense, a quantum field is effectively an infinitely many coupled harmonic oscillators in real space (or uncoupled harmonic oscillators in momentum space).

The Hamiltonian of the scalar field can therefore be written as the sum of all oscillator Hamiltonians

$$\hat{H} = \sum_{n=1}^{\infty}\omega_n\left(\hat{a}_n^\dagger\hat{a}_n + \frac{1}{2}\right). \tag{8}$$

Consider the vacuum state of the field $|0\rangle$, defined to be the state that is annihilated by all $\hat{a}_n$, i.e. $\hat{a}_n |0\rangle = 0$ for all positive integers n. Show that the *vacuum energy expectation value per unit length* is given by

$$E_0(L) = \frac{\pi}{2L^2} \sum_{n=1}^{\infty} n \,. \tag{9}$$

Is this answer problematic?

Solution: The vacuum expectation value is simply

$$\langle 0|\hat{H}|0\rangle = \sum_{n=1}^{\infty} \omega_n \left(\langle 0|\hat{a}_n^\dagger \hat{a}_n|0\rangle + \frac{1}{2} \right) = \frac{\pi}{2L} \sum_{n=1}^{\infty} n \,, \tag{10}$$

and hence the vacuum density is simply $E_0 = \frac{1}{L} \langle 0|\hat{H}|0\rangle$. This is problematic because this sum is formally (naively) divergent. This has to do with the (infinite) zero-point energy of free Minkowski space, since the spacetime is infinite in extent (hence infinitely many oscillators contributing their zero point energy).

(d) Consider the *regularized*[2] vacuum density defined to be

$$E_0(L, \epsilon) \coloneqq \frac{\pi}{2L^2} \sum_{n=1}^{\infty} n e^{-\frac{\epsilon n}{L}} \,. \tag{11}$$

Show that for small ϵ this can be written as a series

$$E_0(L, \epsilon) = \frac{\pi}{2\epsilon^2} - \frac{\pi}{24L^2} + \frac{1}{L^2} O\left(\frac{\epsilon^2}{L^2} \right) \,. \tag{12}$$

Hint: the following may be useful:

$$\sum_n n e^{-nx} = -\frac{\partial}{\partial x} \sum_n e^{-nx} \,, \qquad \frac{e^{-\epsilon/L}}{(1 - e^{-\epsilon/L})^2} = \frac{1}{4 \sinh^2(\frac{\epsilon}{2L^2})} \,, \tag{13}$$

and also

$$\text{cosech}(x) = \frac{1}{x} - \frac{x}{6} + O(x^2) \,. \tag{14}$$

[2]The ϵ takes the role of a UV (high frequency) regulator, i.e. a "UV cut-off", since no experiment can probe arbitrarily high frequency/energy modes of the oscillators.

Solution: Using the first hint, we get

$$\begin{aligned}
E_0(L,\epsilon) &= \frac{\pi}{2L^2}\left(-L\frac{\partial}{\partial\epsilon}\right)\sum_{n=1}^{\infty} e^{-\frac{n\epsilon}{L}} \\
&= -\frac{\pi}{2L}\frac{\partial}{\partial\epsilon}\frac{e^{-\epsilon/L}}{1-e^{-\epsilon/L}} \\
&= \frac{\pi}{2L^2}\frac{e^{-\epsilon/L}}{(1-e^{-\epsilon/L})^2} \\
&= \frac{\pi}{8L^2\sinh^2(\frac{\epsilon}{2L})}\,. \qquad (15)
\end{aligned}$$

In the last equality we used the second hint. Using the third hint, the series expansion reads

$$\begin{aligned}
E_0(L,\epsilon) &= \frac{\pi}{8L^2}\left(\frac{2L}{\epsilon}-\frac{\epsilon}{12L}\right)^2+\frac{1}{L^2}O\left(\frac{\epsilon^2}{L^2}\right) \\
&= \frac{\pi}{2\epsilon^2}-\frac{\pi}{24L^2}+\frac{1}{L^2}O\left(\frac{\epsilon^2}{L^2}\right)\,. \qquad (16)
\end{aligned}$$

(e) Note that for small ϵ, $E_0(L,\epsilon)$ has a divergent piece of order ϵ^{-2}, which corresponds to zero-point energy of the field in "free space". Since free space is infinite in extent and absolute energy cannot be measured (only energy difference), this divergent piece can be subtracted off.

Using the result from part (d), Conclude that as $\epsilon \to 0$, the **Casimir force** between the two plates is given by

$$F = -\frac{\pi}{24L^2}\,, \qquad (17)$$

and hence this force between the plates is indeed attractive.

Hint: recall the relationship between force and work/energy.

Solution: Subtracting the free space energy and take the limit, the vacuum density is

$$\lim_{\epsilon\to 0} E_0(L,\epsilon) = -\frac{\pi}{24L^2}\,. \qquad (18)$$

Note that vacuum density is already of the form $\frac{\Delta E}{\Delta x}$, which is the work done on the plates "by the vacuum". Thus this is the Casimir force. The negative sign implies that this force is attractive.

(f) Let us redo part (d) using a neat method: recall that the *Riemann zeta function* is defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}\,. \tag{19}$$

Using the fact that $\zeta(s) = -\frac{1}{12}$, show that[3]

$$F = -\frac{\pi}{24L^2}\,. \tag{20}$$

Remark: the solution should be short.

Solution: This is the so-called zeta-regularization: we use the fact that

$$\zeta(-1) = \sum_{n=1}^{\infty} n = -\frac{1}{12}\,, \tag{21}$$

which is a result obtained using the so-called analytic continuation in complex analysis (often sensationalized as 1+2+3+... $= -\frac{1}{12}$). Using this, we immediately get the magical result

$$E_0(L) = -\frac{\pi}{24L^2}\,. \tag{22}$$

almost with no effort.

[3] An interested reader will realize that the value of $\zeta(-1)$ is, taken naively, surprising.