3. Schrödinger equation

(a) Using $\partial_\mu = e^i{}_\mu \partial_i$ and the orthogonality condition of $e^i{}_\mu$, one can show that $e_i{}^\mu \partial_\mu = \partial_i$. Thus, it is straightforward to get $\hat{p}_i = -ie_i{}^\mu \partial_\mu$. To get Eq. (6), note that $\hat{p}_i^2 = \delta^{ij} p_i p_j = -(e^{i\nu}\partial_\nu)(e_i{}^\mu \partial_\mu)$. By virtue of the product rule, one simplifies $\hat{p}_i^2$ such that it is equal to the argument in the square bracket of Eq. (6).

(b) It is straightforward to get the first term of Eq. (8) from the definition of inverse metric tensor $g^{\mu\nu}$. For the second term, we note that $g^{\mu\nu} e^i{}_\nu = e^{i\mu}$ and $\Gamma_\mu{}^{\mu\nu} = g^{\mu\lambda}\Gamma_{\mu\lambda}{}^\nu = -g^{\mu\lambda} e^i{}_\lambda \partial_\mu e_i{}^\nu$, thus $\Gamma_\mu{}^{\mu\nu} = -e^{i\mu}\partial_\mu e_i{}^\nu$.

(c) Using $g_{\mu\nu}$, $g^{\mu\nu}$ and $g^{\mu\nu} g_{\nu\lambda} = e^{i\mu} e_i{}^\nu e_{i\nu} e^i{}_\lambda = \delta^\mu{}_\lambda$ given in part (b), the quantity Γ_μ can be shown to take the form of $\Gamma_{\mu\lambda}{}^\lambda$ in Eq. (7). Also, $\Gamma_\mu{}^{\mu\nu} = -e^{i\mu}\partial_\mu e_i{}^\nu$ found in part (b) can be used to show that $\Gamma_\mu{}^{\mu\nu} + \Gamma_\mu{}^{\nu\mu} = -\partial_\mu g^{\mu\nu}$.

(d) We now take advantage of the fact that the derivatives applied to the coordinate transformation $x^i(q)$ commute, thus causing $\Gamma_{\mu\nu}{}^\lambda$ to be symmetric, i.e. $\Gamma_{\mu\nu}{}^\lambda = \Gamma_{\nu\mu}{}^\lambda$. Therefore, $\Gamma_\mu{}^{\nu\mu} = \Gamma^\nu$. With this, we can write $\Gamma_\mu{}^{\mu\nu} = -\partial_\mu g^{\mu\nu} - \Gamma^\mu = -\partial_\mu g^{\mu\nu} - (\partial_\mu \sqrt{g})/\sqrt{g}$. Via product rule of ∂_μ, we get Eq. (10).

(e) Again, via product rule and the result found in part (d), it is easy to see that Δ in Eq. (8) can be written in a more compact form $\Delta = \partial_\mu (g^{\mu\nu}\sqrt{g}\partial_\nu)/\sqrt{g}$. This is the Laplace-Beltrami operator invoked in Eq. (11).

(f) If the Lagrangian coordinates q_i do not merely reparametrize the Euclidean space but specify the points of a general geometry, we cannot proceed as above and derive the Laplace-Beltrami operator by a coordinate transformation of a Cartesian Laplacian. With the canonical quantization rules being unreliable in curvilinear coordinates, we face severe difficulties in quantizing such a system. Fortunately, a large class of non-Cartesian systems allows for a unique quantum-mechanical description on completely different grounds. These systems have the common property that their Hamiltonian can be expressed in terms of the generators of a group of motion in the general coordinate frame. For symmetry reasons, the correspondence principle must then be imposed on the commutators of the group generators rather than upon the Poisson brackets of the canonical variables p and q. Note that the brackets containing two group generators specify the structure of the group, while those containing a generator and a coordinate specify the defining representation of the group in configuration space. The replacement of these brackets by commutation rules constitutes the proper generalization of the canonical quantization from Cartesian to non-Cartesian coordinates (group quantization). The replacement rule will be referred to as the group correspondence principle. The canonical commutation rules in the Euclidean space may be viewed as a special case of the commutation rules between group generators, i.e. of the Lie algebra of the group. In a Cartesian coordinate frame, the group of motion is the Euclidean group containing translations and rotations. The generators of translations and rotations are the momenta and the angular momenta, respectively. According to the group correspondence principle, the Poisson brackets between the generators and the coordinates have to be replaced by commutation rules. Thus, in the Euclidean space, the commutation rules between group generators and coordinates lead to the canonical quantization rules. This is true in particular for systems whose energy depends on generators of the group of motion other than those of translations, for instance on the angular momenta. Then the commutators between the group generators must be used for quantization, rather than the canonical commutators between positions and momenta.

5. Interacting gas

(a) Trivial (use the Hamiltonian given in the partition function).

(b) The partition function now reads

$$\begin{aligned} Z(N,V,T) &= \frac{1}{N!\lambda^{3N}} \int \prod_i d^3r_i \prod_{j<k} (1+f_{jk}) \\ &= \frac{1}{N!\lambda^{3N}} \int \prod_i d^3r_i \left(1 + \sum_{j<k} f_{jk} + \sum_{j<k,l<m} f_{jk}f_{lm} + \cdots\right) \end{aligned}$$

The first term simply gives a factor of the volume V for each integral, so we get V_N. The second term has a sum, each element of which is the same. They all look like

$$\int \prod_{i=1}^{n} d^3r_i \, f_{12} = V^{N-1} \int d^3r \, f(r)$$

where we've simply changed integration variables from r_1 and r_2 to the centre of mass $\vec{R} = (\vec{r}_1 + \vec{r}_2)/2$ and the separation $\vec{r} = \vec{r}_1 - \vec{r}_2$. We do not need to worry about the limits of integration change in the integral over $\vec{r}$, since the integral over $f(r)$ only picks up contributions from atomic size distances, and this is only actually a problem close to the boundaries of the system where it is negligible. There is a term like this for each pair of particles − that is $N(N-1)/2$ such terms. For $N \sim 10^{23}$, it is effectively $N^2/2$. Ignoring terms quadratic in f and higher, the partition function then takes the form of Eq. (7).

(c) Now that we have $\mathcal{Z}$ in terms of f, we use Helmholtz free energy to get the desired expression for p. It is straightforward to integrate the piecewise potential $U(r)$ and obtain Eq. (9). The virial coefficients read $a = 2\pi r_0^3 U_0/3$ and $b = 2\pi r_0^3/3$.

(d) Looking back to the integral in Eq. (8), we see that a long-range force of the form $1/r^n$ will only give rise to a convergent integral for $n \geq 4$. This means that the techniques described above do not work for long-range potentials with fall-off $1/r^3$ or slower (e.g. Coulomb interactions).

7. Gravitational lensing

(a) Trivial (keep only terms in the same order of the perturbation).

(b) Taking $\mu = \alpha = 0$ in $\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\partial_\beta g_{\nu\alpha} + \partial_\alpha g_{\nu\beta} - \partial_\nu g_{\alpha\beta}\right)$, we get

$$\Gamma^0_{0i} = \frac{1}{1+2\Phi}\partial_i\Phi$$

Assuming weak lens, i.e. $\Phi \ll 1$ and keeping term that is in the order of the pertubation, we have $\Gamma^0_{0i} = \partial_i\Phi$. While setting $\alpha = \beta = 0$, we get $\Gamma^i_{00} = \partial^i\Phi$. The other components read

$$\begin{aligned}
\Gamma^i_{jk} &= \frac{1}{2}\delta^{il}\left(\partial_k g_{lj} + \partial_j g_{lk} - \partial_l g_{jk}\right) \\
&= \frac{1}{2}\delta^{ii}\left(\delta^i_j\partial_k\left(1-2\Phi\right) + \delta^i_k\partial_j\left(1-2\Phi\right) - \delta_{jk}\partial_i\left(1-2\Phi\right)\right) \\
&= \delta_{jk}\partial^i\Phi - \delta^i_k\partial_j\Phi - \delta^i_j\partial_k\Phi
\end{aligned}$$

(c) Note that the geodesic equation reads

$$\frac{d^2X^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\nu}\frac{dX^\mu}{d\lambda}\frac{dX^\nu}{d\lambda} = 0$$

Using the null geodesic and taking $\alpha = 0$,

$$\begin{aligned}
\frac{dk^0}{d\lambda} + \frac{dl^0}{d\lambda} + \Gamma^0_{00}\left(\frac{dx^0}{d\lambda}\right)^2 + 2\Gamma^0_{i0}\frac{dx^i}{d\lambda}\frac{dx^0}{d\lambda} &= 0 \\
\frac{dk^0}{d\lambda} + \frac{dl^0}{d\lambda} + 2\frac{d\Phi}{d\lambda}\left(k^0 + l^0\right) &= 0
\end{aligned}$$

Keeping terms in the same order of the pertubation, we have

$$\frac{dl^0}{d\lambda} + 2k\frac{d\Phi}{d\lambda} = 0$$

The other components read

$$\begin{aligned}
\frac{dk^i}{d\lambda} + \frac{dl^i}{d\lambda} + \Gamma^i_{00}\left(\frac{dx^0}{d\lambda}\right)^2 + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} &= 0 \\
\frac{dl^i}{d\lambda} + k^2\partial^i\Phi + \left(\delta_{\alpha\beta}\partial_i\Phi - \delta^i_\beta\partial_\alpha\Phi - \delta^i_\alpha\partial_\beta\Phi\right)\left(k^\alpha + l^\alpha\right)\left(k^\beta + l^\beta\right) &= 0 \\
\frac{dl^i}{d\lambda} + k^2\partial^i\Phi + k_\beta k^\beta g_{ii}\partial^i\Phi - \left(\delta^i_\beta\partial_\alpha\Phi + \delta^i_\alpha\partial_\beta\Phi\right)k^\alpha k^\beta &= 0 \\
\frac{dl^i}{d\lambda} + 2k^2\partial^i\Phi - 2k^i k^\sigma\partial_\sigma\Phi &= 0
\end{aligned}$$

(d) Suppose a light ray starts out into $\hat{z}$-direction and passes a lens at $z = 0$, with impact parameter $b = \sqrt{x^2+y^2}$. For the point mass, $\Phi = -GM/r$ where $r = \sqrt{x^2+y^2+z^2}$. The perpendicular components are

$$\left(\partial_x\Phi, \partial_y\Phi\right) = \frac{GM}{r^3}(x, y) \tag{1.9}$$

Since $\Phi \ll 1$, we expect the deflection angle to be small and hence the total deflection angle can be approximated by the integral over $\dot{l}^i$ along the unperturbed light path:

$$\alpha^i = 2k\int_{\mathcal{P}}\left(\partial^i\Phi - \frac{1}{k^2}k^i k^\sigma\partial_\sigma\Phi\right)d\lambda$$

Obviously $\alpha^x = 0$, since the two terms cancel each other for $\vec{k} = (k, 0, 0)$. Note that $\alpha^y = 2k\int_{\mathcal{P}}\partial^y\Phi\, d\lambda$, and using $x = k\lambda$, we integrate to get $\alpha^y = 4GM/b$.

9. Boson-Fermion correspondence

(a) The grand partition function reads

$$\mathcal{Z} = \sum_i e^{-\beta(E_i - \mu N_i)} \tag{3.1}$$

We rewrite such that N is fixed in the summation and then eventually being summed over all possible N, that is

$$\mathcal{Z} = \sum_{N=0}^{\infty} e^{\beta\mu N} Z_N$$

where Z_N denotes the N-particle canonical partition function

$$Z_N = \sum_i e^{-\beta E_i}$$

where i sums over the systems. We note that the system is in diffusive contact with a reservoir, so the average number is controlled by the chemical potential although the number of bosons is not fixed (open systems). It is impossible to do the calculation in canonical ensemble since the sums cannot be carried out independently of each other, instead we must always make sure that the restriction $\sum_i n_i = N$ is obeyed. If we had no restriction, however, the sums then would factorize into independent sums over each occupation number and they can be computed very easily. This is reason we need to go to the grand canonical ensemble − where the number of particles is not fixed anymore, and hence the restriction is lifted.

(b) With energy given, the grand partition function reads

$$\begin{aligned}\mathcal{Z} &= \sum_{N=0}^{\infty} \left(\exp\left(\beta\mu \sum_j n_j \right) \sum_i \exp\left(-\beta \sum_j n_j(\epsilon_j + U(n_j - 1)) \right) \right) \\ &= \sum_{N=0}^{\infty} \sum_i \prod_j \exp\left(-\beta \left(n_j(\epsilon_j + U(n_j - 1) - \mu \right) \right)\end{aligned}$$

We rewrite $\mathcal{Z}$ again by simply summing over the occupation numbers of every state without restriction:

$$\mathcal{Z} = \prod_j \left(\sum_{n_j} \exp\left(-\beta \left(n_j(\epsilon_j + U(n_j - 1) - \mu \right) \right) \right)$$

(c) The average occupation of the state n_k can be written as follow:

$$\begin{aligned}\langle n_k \rangle &= \frac{1}{\mathcal{Z}} \sum_R n_k \exp\left(-\beta \left(E_R - \mu N_R \right) \right) \\ &= \frac{1}{\mathcal{Z}} \left(\prod_{j \neq k} \sum_{n_j} \exp\left(-\beta \left(n_j(\epsilon_j + U(n_j - 1) - \mu \right) \right) \right) \left(\sum_{n_k} n_k \exp\left(-\beta \left(n_k(\epsilon_k + U(n_k - 1) - \mu \right) \right) \right) \right) \\ &= \frac{\sum_{n_k=0}^{\infty} n_k \exp\left(-\beta \left(n_k(\epsilon_k + U(n_k - 1) - \mu \right) \right)}{\sum_{n_k=0}^{\infty} \exp\left(-\beta \left(n_k(\epsilon_k + U(n_k - 1) - \mu \right) \right)}\end{aligned}$$

(d) For $U = 0$, we have $E = n\epsilon$. It is then obvious that we would recover the Bose-Einstein distribution. To demonstrate this, note that for $U = 0$:

$$\langle n_k \rangle = \frac{\sum_{n_k=0}^{\infty} n_k \exp\left(-\beta \left(n_k(\epsilon_k - \mu \right) \right)}{\sum_{n_k=0}^{\infty} \exp\left(-\beta \left(n_k(\epsilon_k - \mu \right) \right)} = -\frac{1}{\beta} \frac{\partial}{\partial \epsilon_k} \log \left(\sum_{n_k=0}^{\infty} \exp\left(-\beta n_k(\epsilon_k - \mu) \right) \right)$$

which by evaluating the infinite geometric progression gives

$$\langle n_k \rangle = \frac{1}{\beta}\frac{\partial}{\partial \epsilon_k}\log(1-\exp(-\beta(\epsilon_k-\mu))) = \frac{1}{e^{\beta(\epsilon_k-\mu)}-1}$$

(e) In the case $U \to \infty$, we observe that the occupation number n_k has to take $\{0,1\}$ for $\langle n_k \rangle$ in part (c) to make sense. This condition is satisfied by the particles obeying Pauli exclusion principle. So, we would expect $\langle n_k \rangle$ takes the form of Fermi-Dirac distribution. Running the sum only up to $n_k = 1$, we have

$$\langle n_k \rangle = \frac{\sum_{n_k=0}^{1} n_k \exp\left(-\beta\left(n_k(\epsilon_k-\mu)\right)\right)}{\sum_{n_k=0}^{1} \exp\left(-\beta\left(n_k(\epsilon_k-\mu)\right)\right)} = \frac{1}{e^{\beta(\epsilon_k-\mu)}+1}$$

Thus, we see that bosons obey Fermi-Dirac statistics in this limit. These bosons are known as *hard-core bosons*. The limit $U \to \infty$ is known as the *hard-core limit*, corresponding to an infinite repulsive interaction. It has been studied for typical systems with very large repulsive interactions at close range, such as He-4 which can be renormalized (to first order) into hard-core boson.

(f) It is worth noting that the local Hilbert space at every site describes the presence or absence of a single boson and has dimension of two. For spin-1/2 particle, the Hilbert space has the same size, and thus it is possible to provide a map between them two.

(g) One can identify the spin up state for spin-1/2 with the occupied state for a hard core boson and the spin down state with an empty state, i.e. $|\uparrow\rangle \to |1\rangle$ and $|\downarrow\rangle \to |0\rangle$. For hard-core boson, we define the creation $c^\dagger$ and annihilation c satisfying

$$c^\dagger|0\rangle = |1\rangle, \quad c^\dagger|1\rangle = 0, \quad c|0\rangle = 0, \quad c|1\rangle = |0\rangle$$

For spin-1/2 particle, we define the (spin) raising S^+ and lowering S^- operator as $S^+ = S_x + iS_y$ and $S^- = S_x - iS_y$, respectively. They obey

$$S^+|\downarrow\rangle = |1\rangle, \quad S^+|1\rangle = 0, \quad S^-|0\rangle = 0, \quad S^-|1\rangle = |0\rangle$$

Hence, one gets $S^+ \to c^\dagger$ and $S^- \to c$. In evaluating $[c, c^\dagger]$, it is useful to know the fact that $[c, c^\dagger]|0\rangle = |0\rangle$ and $[c, c^\dagger]|1\rangle = -|0\rangle$. It is then easy to see $[c, c^\dagger] = 1 - 2n$. What about that of spin-1/2? We would expect $[c, c^\dagger] = [S^-, S^+]$. The mapping:

$$S^+S^- = S_z + 1/2 \quad \to \quad c^\dagger c = n$$

gives $[S^-, S^+] = 2i[S_x, S_y] = -2S_z = 1 - 2n$. Note that the mapping from spin to fermions or bosons can be rigorously constructed in any dimensions but in general, however, cannot be solved due to strong interactions requirement. However, as we've seen, the case is quite different in $1+1$D $-$ there is a straightforward mapping from bosons to fermions and one could say that they are not fundamentally different. But still, the wavefunction for N hard-core bosons system does not own the antisymmetric property with respect to exchange of two particles that is present in fermionic system.