Solution

a)

The forces on the particle on the left are $-kx_1$ and $-\kappa(x_1 - x_2)$. The forces acting on the particle on the right are $-kx_2$ and $-\kappa(x_2 - x_1)$. Using Newton's second law, one can get

$$m\ddot{x}_1 = -(k+\kappa)x_1 + \kappa x_2,$$

$$m\ddot{x}_2 = \kappa x_1 - (k+\kappa)x_2,$$

which can be arranged into the form of

$$\ddot{\mathbf{x}} = -\frac{1}{m}\mathbf{K}\mathbf{x} \tag{14}$$

where $\mathbf{x} = (x_1, x_2)^T$, and $\mathbf{K} = \begin{pmatrix} k+\kappa & -\kappa \\ -\kappa & k+\kappa \end{pmatrix}$.

b)

Let $\mathbf{x} = \mathbf{C}e^{i\omega t}$, where $\mathbf{C}$ is a column vector, be a solution to Eqn. 14. One gets

$$-\omega^2 \mathbf{C}e^{i\omega t} = -\frac{1}{m}\mathbf{K}\mathbf{C}e^{i\omega t}$$

or, equivalently

$$\frac{1}{m}\mathbf{K}\mathbf{C} = \omega^2\mathbf{C},$$

meaning that $\mathbf{C}$ is an eigenvector of $\frac{1}{m}\mathbf{K}$ with eigenvalue ω^2. One could find the eigenvalues and eigenvectors of $\frac{1}{m}\mathbf{K}$ without too much effort to be

$$\omega_s^2 = \frac{k}{m}, \mathbf{C}_s = (1,1)^T, \text{and}$$

$$\omega_d^2 = \frac{k+2\kappa}{m}, \mathbf{C}_d = (1,-1)^T,$$

where "s" and "d" stand for sum and difference normal modes.

For $\kappa \ll k$, ω_s remains unchanged and $\omega_d \approx \omega_s(1 + \frac{\kappa}{k})$.

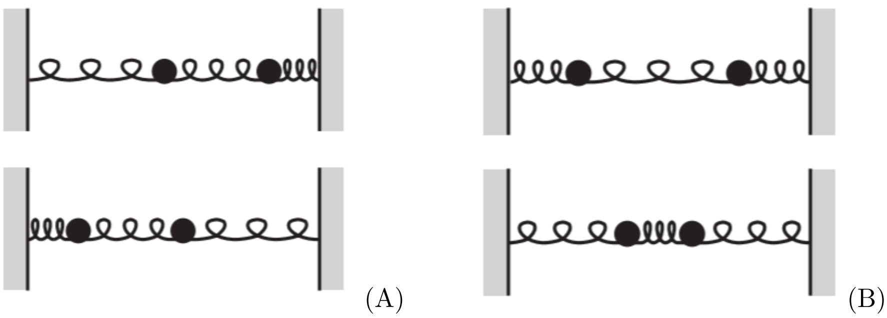

Figure 7: (A) Sum and B difference normal mode of the coupled oscillators.

c)

The general solution to Eqn. 14 is

$$\mathbf{x}(t) = A_1 \mathbf{C}_s e^{i\omega_s t} + A_2 \mathbf{C}_d e^{i\omega_d t}.$$

At $t = 0$, $\mathbf{x}(0) = (A, 0)^T = A_1 \mathbf{C}_s + A_2 \mathbf{C}_d$, giving $A_1 = A_2 = A$ and

$$x_1(t) = A(\cos\omega_s t + \cos\omega_d t) = 2A\cos\left(\frac{\omega_s + \omega_d}{2}t\right)\cos\left(\frac{\omega_d - \omega_s}{2}t\right) \approx 2A\cos(\omega_s t)\cos\left(\frac{\omega_s \kappa t}{2k}\right)$$

$$x_2(t) = A(\cos\omega_s t - \cos\omega_d t) = 2A\sin\left(\frac{\omega_s + \omega_d}{2}t\right)\sin\left(\frac{\omega_d - \omega_s}{2}t\right) \approx 2A\sin(\omega_s t)\sin\left(\frac{\omega_s \kappa t}{2k}\right)$$

Since $\kappa \ll k$, the second terms in x_1 and x_2 oscillate much slower than the first term and can be treated as part of the amplitudes of x_1 and x_2. One can see that the energy is hoping between oscillators as their amplitudes oscillate. Note: the plots are just for demonstration. The students are not required to provide any

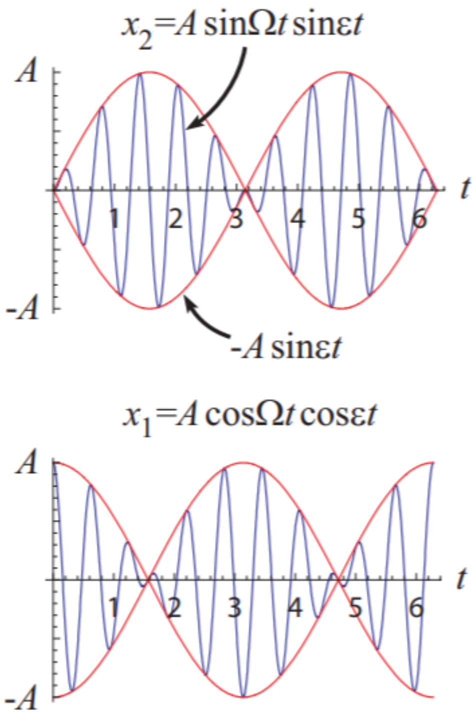

Figure 8: Energy hoping between the oscillators. Here $\Omega = \omega_s$ and $\varepsilon = \frac{\omega_s \kappa}{2k}$.

of these plots.

d)

Using the equations of motion in part (a) and changing k to k_1 and k_2 accordingly, you should get the answer:

$$\ddot{x}_1 + \omega_1^2 x_1 = \omega_c^2 x_2$$

$$\ddot{x}_2 + \omega_2^2 x_2 = \omega_c^2 x_1$$

where $\omega_1^2 = (k_1 + \kappa)/m \approx k_1/m$, $\omega_2^2 = (k_2 + \kappa)/m \approx k_2/m$, and $\omega_c^2 = \kappa/m$ for $\kappa \ll k_1, k_2$.

e)

The first equation in part (d) becomes:

$$\ddot{A} + 2i\omega_1\dot{A} - \omega_1^2 A + \omega_1^2 A = \sqrt{\frac{\omega_1}{\omega_2}}\omega_c^2 B e^{i(\omega_2-\omega_1)t}.$$

Under the slow-varying amplitude assumption, one can drop the $\ddot{A}$ term. The final equation reads

$$i\dot{A} = \frac{\Omega}{2}e^{i\delta t}B.$$

By going through the same process for x_2, one get

$$i\dot{B} = \frac{\Omega}{2}e^{-i\delta t}A,$$

which can be combined with the equation above to give the answer in matrix form.

f)

By substituting $A = ae^{i\frac{\delta}{2}t}$ and $B = be^{-i\frac{\delta}{2}t}$ into the matrix equation in part (e), one get

$$i\begin{pmatrix}\dot{a}+i\frac{\delta}{2}\\ \dot{b}-i\frac{\delta}{2}\end{pmatrix} = \begin{pmatrix}0 & \frac{\Omega}{2}\\ \frac{\Omega}{2} & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix},$$

or equivalently,

$$i\frac{\partial}{\partial t}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}\frac{\delta}{2} & \frac{\Omega}{2}\\ \frac{\Omega}{2} & \frac{-\delta}{2}\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \left[\frac{\delta}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \frac{\Omega}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right]\begin{pmatrix}a\\ b\end{pmatrix} = H\begin{pmatrix}a\\ b\end{pmatrix}.$$

g)

The eigenvalues and eigenvectors of $\begin{pmatrix}\frac{\delta}{2} & \frac{\Omega}{2}\\ \frac{\Omega}{2} & \frac{-\delta}{2}\end{pmatrix}$ are

$$\lambda_\pm = \pm\frac{1}{2}\sqrt{\Omega^2+\delta^2} = \pm\frac{1}{2}\Omega', \text{ and}$$

$$\mathbf{N}_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{1\pm\frac{\delta}{\Omega'}}\\ \pm\sqrt{1\mp\frac{\delta}{\Omega'}}\end{pmatrix}$$

By letting $\frac{\delta}{\sqrt{\delta^2+\Omega^2}} = \cos\theta$, one get

$$\mathbf{N}_+ = \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}; \mathbf{N}_- = \begin{pmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}\end{pmatrix}.$$

By going back, step by step, one get

$$\begin{pmatrix}a(t)\\ b(t)\end{pmatrix} = N_{0+}e^{-i\frac{\Omega'}{2}t}\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix} + N_{0-}e^{i\frac{\Omega'}{2}t}\begin{pmatrix}\sin\frac{\theta}{2}\\ -\cos\frac{\theta}{2}\end{pmatrix};$$

$$\begin{pmatrix}A(t)\\ B(t)\end{pmatrix} = N_{0+}e^{-i\frac{\Omega'}{2}t}\begin{pmatrix}e^{i\frac{\delta}{2}t}\cos\frac{\theta}{2}\\ e^{-i\frac{\delta}{2}t}\sin\frac{\theta}{2}\end{pmatrix} + N_{0-}e^{i\frac{\Omega'}{2}t}\begin{pmatrix}e^{i\frac{\delta}{2}t}\sin\frac{\theta}{2}\\ -e^{-i\frac{\delta}{2}t}\cos\frac{\theta}{2}\end{pmatrix}.$$

Finally,

$$x(t) = \frac{1}{\sqrt{\omega_1}}Re\left\{N_{0+}e^{i(\bar{\omega}-\frac{\Omega'}{2})t}\cos\frac{\theta}{2} + N_{0-}e^{i(\bar{\omega}+\frac{\Omega'}{2})t}\sin\frac{\theta}{2}\right\},$$

$$y(t) = \frac{1}{\sqrt{\omega_2}} Re\left\{ N_{0+} e^{i(\bar{\omega} - \frac{\Omega'}{2})t} \sin\frac{\theta}{2} - N_{0-} e^{i(\bar{\omega} + \frac{\Omega'}{2})t} \cos\frac{\theta}{2} \right\}.$$

For $\delta \ll -\Omega$,

$$\mathbf{N}_+ \approx \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \mathbf{N}_- \approx \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

meaning the $\mathbf{N}_+$ mode majorly consists of x_2 oscillation and the mode $\mathbf{N}_-$ mode majorly consists of x_1 oscillation.

Similarly, for $\delta \gg \Omega$,

$$\mathbf{N}_+ \approx \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \mathbf{N}_- \approx \begin{pmatrix} 0 \\ -1 \end{pmatrix},$$

meaning the $\mathbf{N}_+$ mode majorly consists of x_1 oscillation and the mode $\mathbf{N}_-$ mode majorly consists of x_2 oscillation.

By starting the oscillation in the left most spring with $\omega_1 \ll \omega_2$, the system is in the $\mathbf{N}_-$ mode. Then, by slowly increasing ω_1, the mode of the system remains unchanged until $\omega_2 \ll \omega_1$ which majorly consists of x_2 oscillation. At the end, only the right spring would be oscillating.

Comment: This phenomenon is being used in atomic state transfer and is called adiabatic transfer. Given an atom with ground state $|g\rangle$ and excited state $|e\rangle$ and energy difference E. The transfer process starts by shining a laser beam with frequency $\omega \ll E/\hbar$ at the atom in $|g\rangle$. Then, the laser frequency is slowly ramped up until $\omega \gg E/\hbar$, leaving the atom in $|e\rangle$. To bring the atom from $|e\rangle$ to $|g\rangle$, one can also ramp the laser frequency in a similar fashion (slowly increase the frequency from $\omega \ll E/\hbar$ to $\omega \gg E/\hbar$) or an opposite fashion (slowly decrease the frequency from $\omega \gg E/\hbar$ to $\omega \ll E/\hbar$). The success of state transfer $|g\rangle \to |e\rangle$ is given by the Landau-Zener theory: $P_e = 1 - \exp\left(-\frac{\pi}{2}\frac{\Omega^2}{(d\delta/dt)}\right)$, meaning the slower one ramp the laser frequency, the higher probability that the atom would be in $|e\rangle$. If you are wondering what is the counterpart of the oscillators in this atom + laser system, it is the $|g + \text{laser field}\rangle$ and the $|e + \text{laser field losing one photon}\rangle$.

Solution

a) Amplitude modulation

The electric field after passing through the amplitude modulating device is:

$$\begin{aligned} E_{AM}/E_0 &= \cos(\omega t)\dot{(}0.75 + 0.25\cos(\omega_m t)) = 0.75\cos(\omega t) + 0.25\cos(\omega t)cos(\omega_m t) \\ &= 0.75\cos(\omega t) + \frac{1}{8}(\cos((\omega+\omega_m)t) + \cos((\omega-\omega_m)t)) \end{aligned}$$

The photodetector signal is

$$\begin{aligned} I &= \alpha E_0^2(0.75\cos(\omega t) + \frac{1}{8}(\cos((\omega+\omega_m)t) + \cos((\omega-\omega_m)t)))^2 \\ &= \alpha E_0^2(\frac{3}{16}\cos(\omega t)\cos((\omega+\omega_m)t) + \frac{3}{16}\cos(\omega t)\cos((\omega-\omega_m)t) + \frac{1}{16}\cos((\omega+\omega_m)t)\cos((\omega-\omega_m)t)) \\ &= \alpha E_0^2(\frac{3}{16}\cos(\omega_m t) + \frac{1}{32}\cos(2\omega_m t)). \end{aligned}$$

where we ignore all the oscillations at high frequency. One can see that the photodetector can pick up the modulation frequency and its second harmonic applied via the optical element.

b) Phase and frequency modulation (PM/FM)

The instantaneous angular frequency is defined by

$$\omega_{instant} = \frac{d}{dt}\text{phase} = \frac{d}{dt}(\omega t + \beta\cos(\omega_m t)) = \omega - \beta\omega_m\sin(\omega_m t)$$

By rewriting the phase-modulated field, one obtains

$$\begin{aligned} E_{PM} &= E_0 Re\left\{\exp(i\omega t + i\beta\cos(\omega_m t))\right\} \\ &= E_0 Re\left\{\exp(i\omega t)\exp\left(\frac{i\beta}{2}\left(e^{i\omega_m t} + e^{-i\omega_m t}\right)\right)\right\} \\ &= E_0 Re\left\{\exp(i\omega t)\exp\left(\frac{\beta}{2}\left(ie^{i\omega_m t} - \frac{1}{ie^{i\omega_m t}}\right)\right)\right\} \\ &= E_0 Re\left\{\exp(i\omega t)\sum_{n=-\infty}^{\infty} J_n(\beta)(ie^{i\omega_m t})^n\right\} \\ &= E_0\sum_{n=-\infty}^{\infty} J_n(\beta) Re\left\{\exp(i\omega t)(e^{in(\omega_m t+\pi/2)})\right\} \\ &= E_0\sum_{n=-\infty}^{\infty} J_n(\beta) Re\left\{\exp(i(\omega+n\omega_m)t + in\pi/2))\right\} \\ &= E_0\sum_{n=-\infty}^{\infty} J_n(\beta) cos((\omega+n\omega_m)t + n\pi/2) \end{aligned}$$

For $\beta \ll 1$, $J_n(\beta) \propto \beta^n$. Also, note that $Jn(\beta)$ is real for β real. Therefore, to first order in β,

$$\begin{aligned} E_{PM} &\approx E_0(J_0(\beta)\cos(\omega t) + J_1(\beta)\cos((\omega+\omega_m)t + \pi/2) + J_{-1}(\beta)\cos((\omega-\omega_m)t - \pi/2)) \\ &= E_0(J_0(\beta)\cos\omega t - J_1(\beta)(\sin(\omega+\omega_m)t + \sin(\omega-\omega_m)t)) \end{aligned}$$

Using $J_0(\beta) \approx 1 + O(\beta^2); J_1(\beta) \approx \beta/2 + O(\beta^3)$, one get

$$E_{PM} \approx E_0(\cos\omega t) - \frac{E_0\beta}{2}\left(\sin(\omega+\omega_m)t + \sin(\omega-\omega_m)t\right)$$

One can see that there are three components with frequencies ω, $\omega - \omega_m$, and $\omega + \omega_m$, which is the same as the AM case. However, notice the different in sign of the later two components.

The photodetector signal is

$$
\begin{aligned}
I &= \alpha|E|^2 \\
&= \alpha E_0^2 \left(\cos\omega t - \frac{\beta}{2} \left(\sin(\omega+\omega_m)t + \sin(\omega-\omega_m)t\right) \right)^2 \\
&= \alpha E_0^2 \left(-\beta\cos\omega t \sin(\omega+\omega_m)t - \beta \cos\omega t \sin(\omega-\omega_m)t + O(\beta^2) \right) \\
&= \alpha E_0^2 \left(-\frac{\beta}{2}(\sin(2\omega+\omega_m)t + \sin\omega_m t) - \frac{\beta}{2}(\sin(2\omega-\omega_m)t - \sin\omega_m t) + O(\beta^2) \right) \\
&= \alpha E_0^2 O(\beta^2),
\end{aligned}
$$

which only consists of higher order of β and ω_m. Therefore, one cannot observe the modulation frequency from the photodetector signal but its higher harmonics.

c) Optical cavities

The reflected field phasor is given by

$$
\begin{aligned}
\frac{E_r}{E_i} &= \left[1 - \frac{1}{1 - i\frac{\delta}{\kappa/2}} \right] \\
&= \left[1 - \frac{1 + i\frac{\delta}{\kappa/2}}{1 + \left(\frac{\delta}{\kappa/2}\right)^2} \right] \\
&= \left[1 - \frac{1}{1 + \left(\frac{\delta}{\kappa/2}\right)^2} \right] - i \left[\frac{\frac{\delta}{\kappa/2}}{1 + \left(\frac{\delta}{\kappa/2}\right)^2} \right] \qquad (15)
\end{aligned}
$$

Let's plot the real and imaginary part of E_r

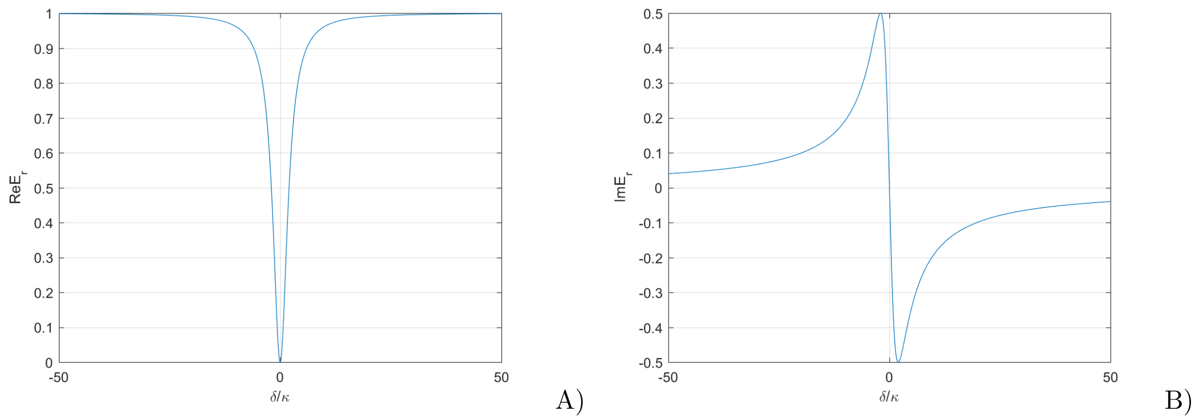

Figure 9: (A) The real part and (B) the imaginary part of the reflected field E_r.

One can see that the real part of E_r is non-zero when $\delta \neq 0$ but it offers no information about whether $\delta > 0$ or $\delta < 0$. The imaginary part of E_r, on the other hand, not only show if the detuning is non-zero but also the sign of the detuning. Therefore, if we can extract the imaginary part of the reflected beam, we can use a feed-back loop to stabilize the frequency of the laser beam to the resonance frequency of the cavity.

d) Optical cavities and amplitude-modulated incident field

The amplitude-modulated field from part (A) is

$$E_{AM}/E_0 = Re\left\{0.75e^{i\omega t} + \frac{1}{8}(e^{(\omega+\omega_m)t} + e^{(\omega-\omega_m)t})\right\}$$

Upon reflection from the cavity, the sidebands are unaffected because they are very far away from the resonance range of the cavity, $|\delta \pm \omega_m| \gg \kappa$. The reflected field is given by

$$\begin{aligned}
\frac{E_{AMr}}{E_0} &= Re\left\{0.75e^{i\omega t}\left[1 - \frac{1}{1+\left(\frac{\delta}{\kappa/2}\right)^2} - i\frac{\frac{\delta}{\kappa/2}}{1+\left(\frac{\delta}{\kappa/2}\right)^2}\right] + \frac{1}{8}(e^{i(\omega+\omega_m)t} + e^{i(\omega-\omega_m)t})\right\} \\
&= 0.75\cos(\omega t)\frac{\left(\frac{\delta}{\kappa/2}\right)^2}{1+\left(\frac{\delta}{\kappa/2}\right)^2} + 0.75\sin(\omega t)\frac{\frac{\delta}{\kappa/2}}{1+\left(\frac{\delta}{\kappa/2}\right)^2} + \frac{1}{8}(\cos((\omega+\omega_m)t) + \cos((\omega-\omega_m)t))
\end{aligned}$$

The photodetector signal of the reflected field is

$$\begin{aligned}
\frac{I}{\alpha E_0^2} &= \frac{3}{16}\frac{\left(\frac{\delta}{\kappa/2}\right)^2}{1+\left(\frac{\delta}{\kappa/2}\right)^2}(\cos(\omega t)\cos((\omega+\omega_m)t) + \cos(\omega t)\cos((\omega-\omega_m)t)) \\
&+ \frac{3}{16}\frac{\frac{\delta}{\kappa/2}}{1+\left(\frac{\delta}{\kappa/2}\right)^2}(\sin(\omega t)\cos((\omega+\omega_m)t) + \sin(\omega t)\cos((\omega-\omega_m)t)) \\
&= \frac{3}{16}\frac{\left(\frac{\delta}{\kappa/2}\right)^2}{1+\left(\frac{\delta}{\kappa/2}\right)^2}(\cos(\omega_m t)) + \frac{3}{32}\frac{\frac{\delta}{\kappa/2}}{1+\left(\frac{\delta}{\kappa/2}\right)^2}(-\sin(\omega_m t) + \sin(\omega_m t)) \\
&= \frac{3}{16}\frac{\left(\frac{\delta}{\kappa/2}\right)^2}{1+\left(\frac{\delta}{\kappa/2}\right)^2}(\cos(\omega_m t)).
\end{aligned}$$

Here, we only keep terms that are slow enough for the photodetector to pick up. The photodetector signal has a component at ω_m of which the amplitude is proportional to the real part of E_r in part (C). Therefore, we cannot use this signal to lock the laser frequency to the cavity resonance frequency.

e) Optical cavities and phase-modulated incident field

The phase-modulated field from part (B) is

$$E_{PM}/E_0 \approx Re\left\{e^{i\omega t} + \frac{i\beta}{2}\left(e^{i(\omega+\omega_m)t} + e^{i(\omega-\omega_m)t}\right)\right\}$$

Upon reflection from the cavity, the sidebands are unaffected because they are very far away from the resonance range of the cavity, $|\delta \pm \omega_m| \gg \kappa$. The reflected field is given by

$$\begin{aligned}
\frac{E_{PMr}}{E_0} &= Re\left\{e^{i\omega t}\left[1 - \frac{1}{1+\left(\frac{\delta}{\kappa/2}\right)^2} - i\frac{\frac{\delta}{\kappa/2}}{1+\left(\frac{\delta}{\kappa/2}\right)^2}\right] + \frac{i\beta}{2}(e^{i(\omega+\omega_m)t} + e^{i(\omega-\omega_m)t})\right\} \\
&= \cos(\omega t)\frac{\left(\frac{\delta}{\kappa/2}\right)^2}{1+\left(\frac{\delta}{\kappa/2}\right)^2} + \sin(\omega t)\frac{\frac{\delta}{\kappa/2}}{1+\left(\frac{\delta}{\kappa/2}\right)^2} - \frac{\beta}{2}(\sin((\omega+\omega_m)t) + \sin((\omega-\omega_m)t))
\end{aligned}$$

The photodetector signal of the reflected field is

$$\begin{aligned}
\frac{I}{\alpha E_0^2} &= -\beta\cos(\omega t)\frac{\left(\frac{\delta}{\kappa/2}\right)^2}{1+\left(\frac{\delta}{\kappa/2}\right)^2}(\sin((\omega+\omega_m)t)+\sin((\omega-\omega_m)t)) \\
&- \beta\sin(\omega t)\frac{\frac{\delta}{\kappa/2}}{1+\left(\frac{\delta}{\kappa/2}\right)^2}(\sin((\omega+\omega_m)t)+\sin((\omega-\omega_m)t)) \\
&= -\frac{\beta}{2}\frac{\left(\frac{\delta}{\kappa/2}\right)^2}{1+\left(\frac{\delta}{\kappa/2}\right)^2}(\sin\omega_m t-\sin\omega_m t)-\frac{\beta}{2}\frac{\frac{\delta}{\kappa/2}}{1+\left(\frac{\delta}{\kappa/2}\right)^2}(\cos\omega_m t+\cos\omega_m t) \\
&= -\beta\frac{\frac{\delta}{\kappa/2}}{1+\left(\frac{\delta}{\kappa/2}\right)^2}\cos\omega_m t
\end{aligned}$$

Here, we neglect all terms that are too fast for the photodetector to pick up. The photodetector signal has only one component at ω_m and that this component is proportional to the imaginary part of E_r in part (C). Therefore, we can use this signal to lock the laser frequency to the cavity resonance.

Comment: To lock the laser frequency, the photodetector signal is then mixed with the signal from the local oscillator that drives the phase-modulating crystal using a mixer. The mixer is an object that takes in two voltages (at the local oscillator LO and input RF ports) and multiplies them to produce a voltage signal out of an output port:

$$V_{mixed} = V_{LO}\times V_{RF}/V_0,$$

with V_0 is a scale factor associated with the mixer, the photodetector signal is sent to the RF-port, and LO-port is supplied with the voltage

$$V_{LO} = V_0\cos(\omega_m t+\phi).$$

After that, the mixed signal is sent through a low-pass filter to get rid of the oscillating terms of frequencies of multiples of ω_m. The constant voltage at the output is

$$V_{out} = -\frac{\beta}{2}\frac{\frac{\delta}{\kappa/2}}{1+\left(\frac{\delta}{\kappa/2}\right)^2}\cos\phi,$$

which can be maximized by shifting the phase of the LO-input signal. The mixer, low-pass filter, and phase shifter are all included in a locking amplifier box commercially available everywhere. The laser is locked to $V_{out}=0$ using a proportional-integral-derivative (PID) controller that takes in V_{out} and send out signals to frequency stabilizing elements of the laser system to stabilize the laser frequency. This whole scheme is called the Pound-Drever-Hall technique discovered by Jan Hall in 1983 when Drever was visiting JILA at CU Boulder and has been used in major experiment, like LIGO, to stabilize lasers.

One more thing worthy of note is that the phase-modulating crystal often provides unwanted amplitude modulation on top of the phase modulation. This amplitude modulation changes with pressure on the electrodes clamping the crystal, temperature, and humidity. From part (D) and (E), you can see that the V_{out} will be shifted around, making locking at $V_{out}=0$ does not correspond to $\delta=0$ and inducing noise to the laser frequency.