PLANCKS SINGAPORE 2021

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- The foolscap papers are for draft work only.
- Write your answers only on the blank a4 paper.
- Start a new question on a new piece of paper.
- Label the question number and page number on the top right corner of each page (e.g. Q1P1, Q1P2, ...)
- Place all sheets containing your answers to each question in one separate envelope. Write question number on the envelope.

1 Casimir effect

It has been experimentally verified that two uncharged parallel conducting plates placed very close to one another (at separation of order μm) experiences an attractive force in vacuum. This result is a *bona fide* prediction of quantum field theory (QFT), as it cannot be explained classically or by non-relativistic quantum mechanics. This question gives a quick exposition to the notion of how a quantum field contains an infinite set of harmonic oscillators in the simple context of a scalar field theory, though the general idea persists in other realistic quantum field theories such as quantum electrodynamics (QED), quantum chromodynamics (QCD), etc. A guiding principle to bear in mind is that many physically interesting aspects of quantum field theory can be traced to the non-trivial nature of its vacuum.

Remark: This question assumes no prior knowledge of QFT. We set $c = \hbar = 1$ throughout, as a PLANCKS member should.

(a) Consider a classical massless scalar field in (1+1)-dimensional spacetime $\phi(t, x)$, subject to Dirichlet boundary condition

$$\phi(t, x = 0) = \phi(t, x = L) = 0\,. \tag{1}$$

Show that if this field satisfies the wave equation

$$\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2}\right)\phi(t, x) = 0\,, \tag{2}$$

the general solution can be written in the form of

$$\phi(t, x) = \sum_{n=-\infty}^{\infty}\left(\alpha_n u_n(t, x) + \beta_n u_n^*(t, x)\right), \tag{3}$$

where $u_n(t, x) = Ne^{-i\omega_n t}\sin\omega_n x$ and N is some normalization constant that does not depend on n, and show that for real scalar field, $\beta_n = \alpha_n^*$.

(b) The **Klein-Gordon inner product** is given by

$$(f, g) := -i\int_0^L \mathrm{d}x\left(f\frac{\partial g^*}{\partial t} - \frac{\partial f}{\partial t}g^*\right). \tag{4}$$

Find the normalization constant N using the orthonormality of the basis functions $(u_m, u_n) = \delta_{m,n}$.

(c) We obtain a quantized scalar field by promoting α_n to an annihilation operator $\hat{a}_n$, and α_n^* to a creation operator $\hat{a}_n^\dagger$. In this sense, a quantum field is effectively an infinitely many coupled harmonic oscillators in real space (or uncoupled harmonic oscillators in momentum space).

The Hamiltonian of the scalar field can therefore be written as the sum of all oscillator Hamiltonians

$$\hat{H} = \sum_{n=1}^{\infty}\omega_n\left(\hat{a}_n^\dagger\hat{a}_n + \frac{1}{2}\right). \tag{5}$$

Consider the vacuum state of the field $|0\rangle$, defined to be the state that is annihilated by all $\hat{a}_n$, i.e. $\hat{a}_n|0\rangle = 0$ for all positive integers n. Show that the *vacuum energy expectation value per unit length* is given by

$$E_0(L) = \frac{\pi}{2L^2}\sum_{n=1}^{\infty} n\,. \tag{6}$$

(d) Consider the *regularized*[1] vacuum density defined to be

$$E_0(L,\epsilon) := \frac{\pi}{2L^2}\sum_{n=1}^{\infty} n e^{-\frac{\epsilon n}{L}}\,. \tag{7}$$

Show that for small ϵ this can be written as a series

$$E_0(L,\epsilon) = \frac{\pi}{2\epsilon^2} - \frac{\pi}{24L^2} + \frac{1}{L^2}O\left(\frac{\epsilon^2}{L^2}\right)\,. \tag{8}$$

Hint: the following may be useful,

$$\sum_n n e^{-nx} = -\frac{\partial}{\partial x}\sum_n e^{-nx}\,, \qquad \frac{e^{-\epsilon/L}}{(1-e^{-\epsilon/L})^2} = \frac{1}{4\sinh^2(\frac{\epsilon}{2L^2})}\,, \tag{9}$$

and also

$$\text{cosech}(x) = \frac{1}{x} - \frac{x}{6} + O(x^2)\,. \tag{10}$$

(e) Note that for $E_0(L,\epsilon)$ has a divergent piece of order ϵ^{-2} for small ϵ, which corresponds to zero-point energy of the field in "free space". Since free space is infinite in extent and absolute energy cannot be measured (only energy difference), this divergent piece can be subtracted off. Using the result from part (d), show that the *Casimir force* between the two plates as $\epsilon \to 0$ reads

$$F = -\frac{\pi}{24L^2}\,, \tag{11}$$

and hence this force between the plates is indeed attractive.

Hint: recall the relationship between force and work/energy.

(f) Let us redo part (d) using a neat method: recall that the *Riemann zeta function* is defined by

$$\zeta(s) := \sum_{n=1}^{\infty}\frac{1}{n^s}\,, \quad \text{Re}(s) > 1. \tag{12}$$

Via analytic continuation of the sum, and using the fact that $\zeta(-1) = -1/12$, show that

$$F = -\frac{\pi}{24L^2}\,. \tag{13}$$

Erickson Tjoa - University of Waterloo

[1] The ϵ takes the role of a UV (high frequency) regulator, i.e. a "UV cut-off", since no experiment can probe arbitrarily high frequency/energy modes of the oscillators.

2 Classical Information Communication

In classical communication, information is sent via modulated electromagnetic waves, and this technology one of the main drivers of the current information age. With the ability to have ever greater control of quantum systems now, it is interesting to explore if using quantum systems would provide an advantage over classical communication. In general, the state of a n-dimensional ($n = 2$ for qubits) quantum system can be expressed as a density matrix, ρ, which is a $n \times n$ matrix with two properties: it has trace 1, and positive eigenvalues. For a system where we prepare states $|\psi_i\rangle$ with probability p_i, we can express the density matrix of the output state as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \tag{1}$$

(a) [2 points] Suppose that we are given a device that can randomly generate any pure qubit ($n = 2$) state with equal probability. Calculate the density matrix of the output state from this device.

We can try to communicate a message $i \in \{1, \ldots, m\}$, (we assume i to appear with uniform probability $p_i = 1/m$) by encoding them as a state on a quantum system, ρ_i. For such an encoding scheme, the amount of classical bits one can communicate successfully via a noiseless channel (more accurately, the mutual information) is bounded by the Holevo quantity:

$$\chi(\rho) = S(\rho) - \sum_i p_i S(\rho_i) \ , \tag{2}$$

where $S(\rho)$ is the von-Neumann entropy. The von-Neumman entropy is a quantum version of the classical entropy, and is defined as

$$S(\rho) = -\sum_j \lambda_j \log_2 \lambda_j \ , \tag{3}$$

where λ_j is the eigenvalue of ρ.

(b) [4 points] Using the properties of a density matrix, prove that the Holevo quantity for any encoding on a n-dimensional quantum system is bounded by $\log_2 n$. Furthermore, prove that this bound is tight.

Setting $n = 2$ shows that using a single qubit confers no additional advantage in classical communication over using a classical bit, but things change if we allow the communicating parties to share an entangled state.

(c) Suppose two parties, A and B, share a two-qubit entangled state $|\psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B)$, with A holding onto the first qubit and B holding onto the second qubit.

(i) [1 point] List the unitaries that A can perform on her qubit in $|\psi^-\rangle_{AB}$ to obtain states $|\psi^-\rangle_{AB}$, $|\psi^+\rangle_{AB}$, $|\phi^+\rangle_{AB}$, and $|\phi^-\rangle_{AB}$.

Note:
$|\psi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)$
$|\phi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B - |1\rangle_A|1\rangle_B)$
$|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$

(ii) [2 points] Propose a method for A to send two classical bits to B if A and B has an entangled state $|\psi^-\rangle_{AB}$.

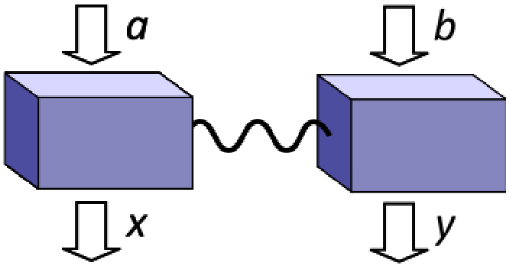

Figure 1: Schematic of a PR box.

We can see that using entanglement, A can convey two bits of classical information to B by sending one qubit. However, there are stronger resources beyond quantum theory (hypothetical ones) that can perform even more powerful communication tasks, and one such resource is the PR box. A PR box (see Figure 1) held by A and B takes in input bit a from A and input bit b from B, and outputs bit x to A and bit y to B following the probability distribution

$$\Pr[x, y|a, b] = \begin{cases} 0.5 & x \oplus y = a \times b \\ 0 & \text{otherwise} \end{cases}, \tag{4}$$

where $\oplus$ is addition modulo 2 (XOR). Suppose we have a communication task where A has two bits of information, k_1, and k_2, and B can choose to learn either k_1 or k_2. If B is not allowed to send any messages to A, it is clear with that A must send over the information of both bits (with one qubit + entanglement, or two classical bits) for B to choose his desired bit at some later time. If a PR box is shared between A and B, this can be achieved using only a single classical bit. To achieve the communication, A first inputs $a = k_0 \oplus k_1$ into the PR box to obtain output x. A then sends a single bit $m = k_0 \oplus x$ to Bob.

(d) [1 point] Explain how B can obtain his desired bit k_j from m and the PR box.

Kon Wen Yu - National University of Singapore

3 Schrödinger equation

The extension of the quantum-mechanical formalism to systems described by a set of general Lagrange coordinates $q_1, \ldots, q_N$ is not straightforward. Note that the correspondence principle are sufficient to quantize the system[2] only in the special case that $q_i\,(i = 1, \ldots, N)$ represent merely a curvilinear reparametrization of a D-dimensional Euclidean space parametrized by x^i. The number N of coordinates is then equal to the dimension D, and a variable change from x^i to q_j in the Schrödinger equation leads to the correct quantum mechanics. It will be useful to label the curvilinear coordinates by Greek superscripts, and write q^μ instead of Latin subscripts in q_j. This will help us to write all ensuing equations in a form that is manifestly covariant under coordinate transformations. For the Cartesian coordinates, we shall use Latin indices alternatively as sub- or superscripts. The coordinate transformation $x^i = x^i(q^\mu)$ implies the relation between the derivatives $\partial_\mu = \partial/\partial q^\mu$ and $\partial_i = \partial/\partial x^i$:

$$\partial_\mu = e^i{}_\mu(q)\partial_i \ , \tag{1}$$

with the transformation matrix,

$$e^i{}_\mu(q) = \partial_\mu x^i(q) \tag{2}$$

called basis D-ad e.g. triad in 3 dimensions, tetrad in 4 dimensions, etc. Assuming it exists, let $e_i{}^\mu(q) = \partial q^\mu/\partial x^i$ be the inverse matrix called the reciprocal $D-$ad, satisfying with $e^i{}_\mu$ the orthogonality and completeness relations:

$$e^i{}_\mu\, e_i{}^\nu = \delta_\mu{}^\nu, \qquad e^i{}_\mu\, e_j{}^\mu = \delta^i{}_j \tag{3}$$

where $\delta^a{}_b = 1$ if $a = b$ and 0 if otherwise. In this question, we adopt Einstein summation convention in which repeated indices are summed over, e.g

$$\sum_{i=1}^{3} b_i c^i = b_1 c^1 + b_2 c^2 + b_3 c^3 = b_i c^i \tag{4}$$

Here the superscript in c^i denotes the i-th component of $\mathbf{c}$. We set $\hbar = 1$ throughout.

(a) [1 point] Show that the curvilinear transform of the Cartesian momentum operators read

$$\hat{p}_i = -i\, e_i{}^\mu(q)\partial_\mu \tag{5}$$

and that the corresponding Hamiltonian operator of a free-particle with mass M takes the form

$$\hat{H}_0 = -\frac{\Delta}{2M} = -\frac{1}{2M}\left[e^{i\mu} e_i{}^\nu \partial_\mu \partial_\nu + (e^{i\mu}\partial_\mu e_i{}^\nu)\partial_\nu\right] . \tag{6}$$

The quantity Δ is known as the Laplacian.

(b) [2 points] We introduce the metric tensor $g_{\mu\nu}(q) := e_{i\mu} e^i{}_\nu(q)$ and its inverse $g^{\mu\nu}(q) = e^{i\mu} e_i{}^\nu(q)$ defined by $g^{\mu\nu} g_{\nu\lambda} = \delta^\mu{}_\lambda$. The affine connection reads

$$\Gamma_{\mu\nu}{}^\lambda(q) = -e^i{}_\nu(q)\, \partial_\mu e_i{}^\lambda(q) \ . \tag{7}$$

Show that the Laplacian now takes the form

$$\Delta = g^{\mu\nu}(q)\partial_\mu\partial_\nu - \Gamma_\mu{}^{\mu\nu}(q)\partial_\nu \ , \tag{8}$$

with $\Gamma_\mu{}^{\lambda\nu}$ being defined as the contraction $\Gamma_\mu{}^{\lambda\nu} = g^{\lambda\kappa}\, \Gamma_{\mu\kappa}{}^\nu$.

[2] The correspondence principle is the rule of obtaining Hamiltonian operator in the Schrödinger equation from the classical Hamiltonian by simply substituting $\mathbf{x} \to \hat{\mathbf{x}}$ and $\mathbf{p} \to -i\partial_{\mathbf{x}}$.

(c) [2 points] The infinitesimal volume element is given by $d^D x = \sqrt{g}\, d^D q$, where $g(q) = \det(g_{\mu\nu}(q))$ denotes the determinant of the metric tensor. With this determinant, we form the quantity

$$\Gamma_\mu := \frac{1}{\sqrt{g}} \left(\partial_\mu \sqrt{g}\right) = \frac{1}{2} g^{\lambda\kappa} \left(\partial_\mu g_{\lambda\kappa}\right) \ . \tag{9}$$

Show that $\Gamma_\mu = \Gamma_{\mu\lambda}{}^{\lambda}$ and $\Gamma_\mu{}^{\mu\nu} = -\partial_\mu g^{\mu\nu} - \Gamma_\mu{}^{\nu\mu}$.

(d) [1 point] Argue why $\Gamma_\mu{}^{\nu\mu} = \Gamma^\mu$ and show that

$$\Gamma_\mu{}^{\mu\nu} = -\frac{1}{\sqrt{g}} \left(\partial_\mu g^{\mu\nu} \sqrt{g}\right) \ . \tag{10}$$

(e) [2 points] The result in part (d) allows us to write Eq. (8) in a more compact form. In differential geometry, this form is known as the Laplace-Beltrami operator. Show that this compact form of Λ allows the Schrödinger equation in curvilinear coordinates to be written as

$$\left[-\frac{1}{2M\sqrt{g}} \partial_\mu g^{\mu\nu} \sqrt{g} \partial_\nu + V(\mathbf{x}(q)) \right] \Psi(q,t) = i\partial_t \Psi(q,t) \ , \tag{11}$$

(f) [2 points] If the Lagrangian coordinates q_i do not merely reparametrize the Euclidean space, but also specify the points of a general geometry, we cannot proceed as above to get Eq. (11) by a coordinate transformation of a Cartesian Laplacian. Explain why.

Hint: think about Poisson brackets, commutation rules, and group generators in Cartesian and non-Cartesian coordinate frames.

Farisan Dary - Nanyang Technological University

4 AM, FM, PM, and Optical Cavity

(a) [2 points] **Amplitude Modulation (AM)**
Given a laser field with electric field strength $E = E_0 \cos(\omega t)$ passing through an optical element that partly modulates the electric field strength of the laser with

$$\frac{1}{4}\left(3 + \cos(\omega_m t)\right) \ , \tag{1}$$

where $\omega_m \ll \omega$. This optical element could be a crystal that changes the polarization axis of the input field depending on the voltage applied across the crystal and a polarizer at the end. Show that the resulting field has three components oscillating at $\omega - \omega_m$, ω, and ω_m.

Suppose we shine this amplitude-modulated field on a fast photodetector, what would be the frequency of the photodetector signal? Given that the photodetector detect the amplitude squared[3] of the input field with some proportional coefficient: $I = \alpha |E|^2$, and the laser frequency ω (often in the 100 THz) is too fast for the photodetector to pick up.

(b) [2 points] **Phase and frequency modulation (PM/FM)**
Imagine now that the input field is modulated by passing light through another crystal whose index of refraction changes when a voltage is applied across the crystal. The applied voltage is at frequency $\omega_m \ll \omega$ and produce a phase shift of $\beta \ll 1$ such that

$$E_{\mathrm{PM}} = E_0 \cos(\omega t + \beta \cos(\omega_m t)) \ . \tag{2}$$

Show that the instantaneous frequency of the signal is

$$\omega_{\mathrm{instant}} = \omega - \beta \omega_m \sin(\omega_m t) \ , \tag{3}$$

which is the same as modulating the frequency of the signal. This explains that PM and FM are closely related. Now, expand the phase modulated signal into its Fourier components to the first order in β. If you shine this phase-modulated laser field on the photodetector given above, what would be the frequency of the photodetector signal to first order in β?

Hint: you may find these expressions useful

$$e^{x(t-t^{-1})/2} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n \ , \tag{4}$$

where $J_n(x)$ are the Bessel functions of the first kind given by

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+n+1)} \left(\frac{x}{2}\right)^{2m+n} , \quad n \geq 0 \ . \tag{5}$$

Also, another property that you may find useful is $J_{-n}(x) = (-1)^n J_n(x)$.

(c) [2 points] **Optical cavities**
An optical cavity is an arrangement of mirrors that forms a standing wave within its interior. We consider the simplest optical cavity made of two curved mirror. Near resonance, the circulating electric field inside of the cavity is given by the complex phasor E_c:

$$E_c = \frac{E_i}{1 - i\frac{\delta}{\kappa/2}} \ , \tag{6}$$

[3] Actually, the photodetector detects the photon number rather than the amplitude squared of the input field. This difference only matters for low and single photon experiment.

where E_i is the incident field, $\delta = \omega - \omega_c$ is the difference of the incident field frequency to the cavity resonance frequency (also called detuning), and κ is the decay rate of energy stored within the cavity.

Besides, there is a reflected field from the cavity when one tries to couple light to the cavity. In a high mirror reflection limit, the reflected field is given by

$$E_r = E_i \left[1 - \frac{1}{1 - i\frac{\delta}{\kappa/2}} \right] \ . \tag{7}$$

Plot the real and imaginary part of the reflected field against δ/κ from $(\delta/\kappa) \ll -1$ to $(\delta/\kappa) \gg 1$ and discuss which one would be useful to tell that the incident field frequency is higher, equal, or lower than the resonance frequency of the cavity?

(d) [2 points] **Optical cavities and amplitude-modulated incident field**

Let's couple the amplitude-modulated field from part (a) into this optical cavity and look at the reflected field. For this question, we are only interested in the regime when $\omega_m \gg \kappa \gg |\delta|$ so that the sidebands of the AM field reflect totally off the cavity. If we detect the reflected field with a photodetector, do we see a frequency component at ω_m? Which part of the reflected field of part (c) (real or imaginary) is included in this signal? Can we use this signal to lock our laser frequency to the cavity resonance frequency?

(e) [2 points] **Optical cavities and phase-modulated incident field**

Now, let's couple the phase-modulated signal from part (b) into this optical cavity and look at the reflected field. For this question, we are only interested in the regime when $\omega_m \gg \kappa \gg |\delta|$ so that the sidebands of the PM field reflect totally off the cavity. If we detect the reflected field with a photodetector, do we see a frequency component at ω_m? Which part of the reflected field of part (c) (real or imaginary) is included in this signal? Can we use this signal to lock our laser frequency to the cavity resonance frequency?

Tin Nghia Nguyen - University of Colorado at Boulder

5 Interacting gas

In introductory statistical mechanics, we always treat the gas particles moving around, unaware of each other. Things will get much more interesting if we take into account the interactions. In this question, we use a simple approximation scheme to understand the effects of interactions between particles, particularly on *monoatomic gas*. The corrections to the equation of state of ideal gas are often expressed in terms of a density expansion, known as *the virial expansion*:

$$\frac{p}{k_\mathrm{B}T} = \frac{N}{V} + B_2(T)\frac{N^2}{V^2} + B_3(T)\frac{N^3}{V^3} + \cdots \ , \tag{1}$$

where the functions $B_j(T)$ are known as the virial coefficients. Our main goal is then to compute the virial coefficients starting from the underlying potential energy $U(r)$ between two neutral atoms separated by a distance r. We restrict ourselves to

$$U(r) = \begin{cases} \infty & r < r_0 \\ -U_0(r_0/r)^6 & r \geq r_0 \end{cases} , \tag{2}$$

which incorporates a hard-core repulsion that forbids the particles to get closer than a certain fixed distance. One can write the Hamiltonian of the gas as follow:

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i>j} U(r_{ij}) \ , \tag{3}$$

where $r_{ij} = |\vec{r}_i - \vec{r}_j|$ is the separation between particles. The restriction $i > j$ on the final sum ensures that we sum over each pair of particles exactly once.

(a) Show that the partition function $Z(N, V, T)$ reads

$$Z(N,V,T) = \frac{1}{N!\lambda^{3N}} \int \prod_i d^3r_i e^{-\beta \sum_{j<k} U(r_{jk})} \ , \tag{4}$$

where $\lambda = \left(2\pi\hbar^2/mk_\mathrm{B}T\right)^{1/2}$ is the thermal wavelength and $\beta = 1/(k_\mathrm{B}T)$.

Hint: you may take the following as a starting point,

$$Z(N,V,T) = \frac{1}{N!}\frac{1}{(2\pi\hbar)^{3N}} \int \prod_{i=1}^{N} d^3p_i d^3r_i e^{-\beta H} \ . \tag{5}$$

(b) It is not so useful to expand the exponential term in Eq. (4) as an expansion parameter. Instead, we consider the Mayer f function $f(r) = e^{-\beta U(r)} - 1$ as an expansion parameter. We further define $f_{ij} = f(r_{ij})$ such that

$$Z(N,V,T) = \frac{1}{N!\lambda^{3N}} \int \prod_i d^3r_i \prod_{j>k} (1 + f_{jk}) \ . \tag{6}$$

Hence, show that the above expression can be written as follow:

$$Z(N,V,T) = Z_\mathrm{ideal} \left(1 + \frac{N}{2V} \int d^3r\, f(r) + \cdots \right)^N \ , \tag{7}$$

where $Z_\mathrm{ideal} = V^N/(N!\lambda^{3N})$. Note that the density of the gas is small.

(c) By ignoring terms quadratic in f and higher, show that

$$\frac{pV}{Nk_\mathrm{B}T} = 1 - \frac{N}{2V} \int d^3r\, f(r) \ , \tag{8}$$

and using $U(r)$ given in Eq. (2), show that in the limit $\beta U_0 \ll 1$:

$$\int d^3r\, f(r) = \frac{4\pi r_0^3}{3}\left(\frac{U_0}{k_{\mathrm{B}}T} - 1\right) \ . \tag{9}$$

Hence, determine the virial coefficients.

Hint: in terms of the partition function Z, Helmholtz free energy F is given by $F = -k_{\mathrm{B}}T \log Z$. Related thermodynamic quantities can be deduced from the fundamental thermodynamic relation

$$dF = -S\,dT - P\,dV + \mu\,dN \ . \tag{10}$$

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(d) Can we use Eq. (8) for any potential $U(r)$ between atoms? State the limitations, if any.

Farisan Dary - Nanyang Technological University

6 Chimney Physics

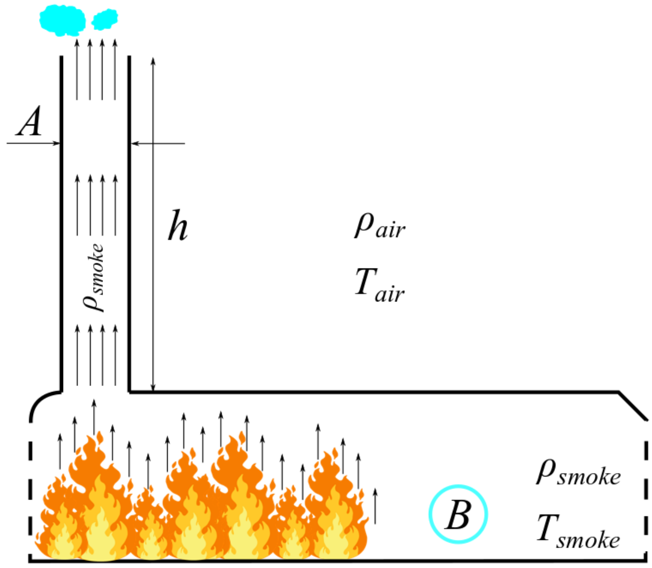

Figure 2: Sketch of a chimney of height h with a furnace at temperature T_{smoke}.

Gaseous products of burning are released into the atmosphere of temperature T_{air} through a high chimney of cross-section A and height h (see Fig.2). The temperature inside the furnace is T_{smoke} and the volume of gases produced per unit time in the furnace is B. Throughout this question, we assume the following:

- the velocity of the gases in the furnace is negligibly small.
- the density of the gases (smoke) does not differ from that of the air at the same temperature and pressure; while in the furnace, the gases can be treated as ideal.
- the pressure of the air changes with height in accordance with the hydrostatic law and the change of the density of the air with height is negligible.
- the flow of gases satisfies the Bernoulli equation which states

$$\frac{1}{2}\rho v^2(z) + \rho g z + p(z) = \text{ const },\tag{1}$$

  i.e. it is conserved at all points in the flow. Here ρ, $v(z)$, $p(z)$, and z denote the density of the gas, velocity, pressure, and height, respectively.
- the change in the gas density is negligible throughout the chimney.

(a) [0.5 point] What is the minimal height of the chimney needed in order for the chimney to function efficiently such that it can release all of the produced gas into the atmosphere? Express your result in terms of B, A, T_{air}, $g = 9.81\,\text{m/s}^2$, $\Delta T = T_{\text{smoke}} - T_{\text{air}}$. **Important**: in all subsequent tasks, assume that this minimal height is the height of the chimney.

(b) [0.5 point] Assume that two chimneys are built to serve exactly the same purpose. Their cross sections are identical, but are designed to work in different parts of the world: one in cold regions (designed to work at an average atmospheric temperature of $-30°$C) and the other in warm regions (designed to work at an average atmospheric temperature of $30°$C). The temperature of the furnace is 400 °C. It was calculated that the height of the chimney designed to work in cold regions is 100 m. How high is the other chimney?

(c) [1 point] How does the velocity of the gases vary along the height of the chimney? Provide a graph that shows how the velocity varies along the height of the chimney with the assumption that its cross-section does not change along the height. Indicate the point where the gases enter the chimney.

(d) [1 point] How does the pressure of the gases vary along the height of the chimney?

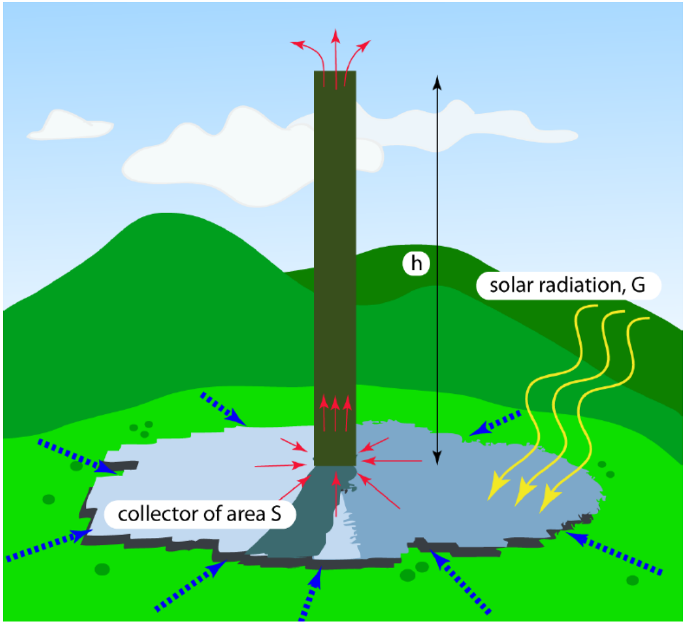

Figure 3: Sketch of a solar power plant.

The flow of gases in a chimney can be used to construct a particular kind of solar power plant (solar chimney), as illustrated in Fig.3. The Sun heats the air underneath the collector of area S with an open periphery to allow the undisturbed inflow of air. As the heated air rises through the chimney (thin solid arrows), new cold air enters the collector from its surrounding (thick dotted arrows), enabling a continuous flow of air through the power plant. The flow of air through the chimney powers a turbine, resulting in the production of electrical energy. The energy of solar radiation per unit time per unit of horizontal area of the collector is G. Assume that all that energy can be used to heat the air in the collector (the mass heat capacity of the air is c, and one can neglect its dependence on the air temperature). We define the efficiency of the solar chimney as the ratio of the kinetic energy of the gas flow and the solar energy absorbed in heating of the air prior to its entry into the chimney.

(e) [1 point] What is the efficiency of the solar chimney power plant?

(f) [1 point] Provide a graph that shows how the efficiency of the chimney varies with its height.

The prototype chimney built in Manzanares, Spain, had a height of 195 m, and a radius 5 m. The collector is circular with diameter of 244 m. The specific heat of the air under typical operational conditions of the prototype solar chimney is $1012\,\mathrm{J/(K\,kg)}$, the density of the hot air is about $0.9\,\mathrm{kg/m^3}$, and the typical temperature of the atmosphere $T_{\mathrm{air}} = 295\,\mathrm{K}$. In Manzanares, the solar power per unit of horizontal surface is typically $150\mathrm{W/m^2}$ during a sunny day.

(g) [1 point] What is the efficiency of the prototype power plant? Write down the numerical estimate.

(h) [1 point] How much power could be produced in the prototype power plant?

(i) [1 point] How much energy could the power plant produce during a typical sunny day?

(j) [1 point] How large is the rise in the air temperature as it enters the chimney (warm air) from the surrounding (cold air)? Write the general formula and evaluate it for the prototype chimney.

(k) [1 point] What is the mass flow rate of air through the system?

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7 Gravitational lensing

Gravitational lensing refers to a phenomenon in which light rays are being deflected by gravitational fields of massive objects, thus acting as lenses. The bending of light was one of the first important experimental signatures of General Relativity. In particular, it continues to play crucial roles in dark matter or black hole detection. The basic idea is that dark matter, though invisible in many ways, could be seen through its gravitational influence on visible objects and, in particular, by its bending of light emitted by other luminous objects. One of the most compelling pieces of evidence for dark matter comes from the Bullet Cluster formed from the merger of two galaxy clusters.

In this question, we deal with a simple exposition of the topic. We assume that the lens is weak, that is, its Newtonian gravitational potential Φ is much smaller than c^2. This approximation is valid in virtually all cases of astrophysical interest. For instance, a galaxy cluster has gravitational potential $|\Phi| < 10^{-4}c^2 \ll c^2$. From now on, we set $c = 1$ and begin with the following line element that describes the spacetime for static Newtonian gravitational sources:

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2+dy^2+dz^2) \ , \tag{1}$$

where Φ is the Newtonian gravitational field. In this question, we adopt Einstein summation convention where repeated indices are implicitly summed over. For example,

$$ds^2 = \sum_{i=1}^{3}\sum_{j=1}^{3} g_{ij}dx^i dx^j = g_{ij}dx^i dx^j \tag{2}$$

where g_{ij} is known as the metric tensor that describes the spacetime geometry. Assuming that we live in $3+1$D (1 temporal dimension and 3 spatial dimensions), we write $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ for $\mu, \nu \in \{0,1,2,3\}$ with $x^0 = t$ and $x^i = \{x, y, z\}$. Greek indices are used when we include time and Latin indices when we're just describing space. Note that we can raise or lower the indices of a tensor using the metric tensor $g_{\mu\nu}$ or its inverse. For examples: $g_{\mu\tau}A^{\alpha\beta\tau} = A^{\alpha\beta}{}_{\mu}$ and $g^{\mu\nu}\partial_\nu = \partial^\mu$.

(a) [2 points] Writing $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu}$ is the perturbation. We consider the null geodesics:

$$g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0 \tag{3}$$

on this background with λ parametrizes the path of photons and calculate the deflection of light using first-order perturbation theory. We decompose the geodesic $x^\mu(\lambda) = x^{(0)\mu}(\lambda) + x^{(1)\mu}(\lambda)$ and solve for the perturbation $x^{(1)\mu}(\lambda)$ by performing calculations on the flat background. Show that by defining the wavevectors:

$$k^\mu \equiv \frac{dx^{(0)\mu}(\lambda)}{d\lambda}, \quad l^\mu \equiv \frac{dx^{(1)\mu}(\lambda)}{d\lambda} \ , \tag{4}$$

the null geodesics yields $\eta_{\mu\nu}k^\mu k^\nu = 0$ and $h_{\mu\nu}k^\mu k^\nu + 2\eta_{\mu\nu}k^\mu l^\nu = 0$.

(b) [2 points] Show that the affine connections read

$$\Gamma^0_{0i} = \partial_i\Phi, \quad \Gamma^i_{00} = \partial^i\Phi, \quad \Gamma^i_{jk} = \delta_{jk}\partial^i\Phi - \delta^i_k\partial_j\Phi - \delta^i_j\partial_k\Phi \ . \tag{5}$$

Hint: they can be obtained from

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\partial_\beta g_{\nu\alpha} + \partial_\alpha g_{\nu\beta} - \partial_\nu g_{\alpha\beta}\right) \ . \tag{6}$$

(c) [3 points] From the geodesic equation, show that

$$\frac{dl^0}{d\lambda} + 2k\frac{d\Phi}{d\lambda} = 0, \quad \text{and} \quad \frac{dl^i}{d\lambda} + 2k^2\partial^i\Phi - 2k^i\vec{k}\cdot\vec{\nabla}\Phi = 0 \ . \tag{7}$$

Hint: let a one-dimensional curve in spacetime be parametrized by λ, i.e. $X = X(\lambda)$. The geodesic equation reads

$$\frac{d^2X^\alpha}{d\lambda^2} + \Gamma^\alpha_{\rho\nu}\frac{dX^\nu}{d\lambda}\frac{dX^\rho}{d\lambda} = 0 \ . \tag{8}$$

If we the proper time as our curve parameter, solving this second-order differential equations gives the path that extremizes the proper time between the two points.

(d) [3 points] The null geodesic is deflected by the gravitational field of the point mass M such that it is no longer a straight line. We now wish to derive the deflection angle α as a function of the mass M and impact parameter. Noting

$$\alpha^i = -\frac{1}{k}\int_{\mathcal{P}} \frac{dl^i}{d\lambda} d\lambda \ , \tag{9}$$

where the integral is carried out on the background path $\mathcal{P}$. Note that k^i are constants, taking $x^{(0)i} = 0$ at $\lambda = 0$, we can write $x^{(0)i} = k^i\lambda$.

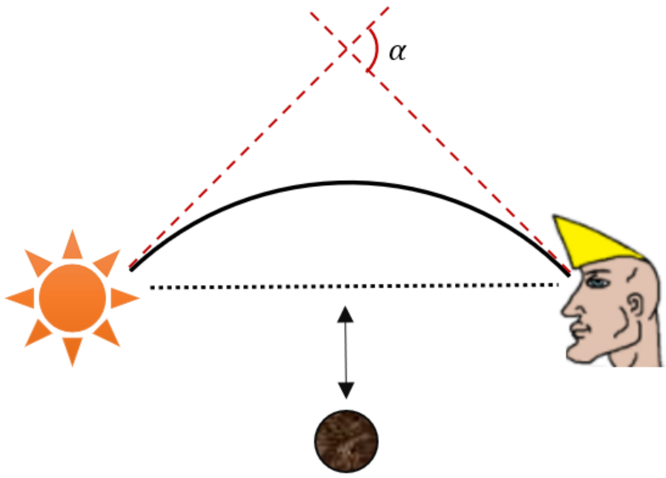

Figure 4: The photon path being deflected by the gravitational field of the point mass.

Consider the case of a point mass with the background path along the x-direction, i.e. $k = (k, 0, 0)$, $x = k\lambda$, $y = b$ and we also choose $z = 0$. Suppose the gravitational potential reads

$$\Phi = -\frac{GM}{\sqrt{x^2 + y^2}} \ . \tag{10}$$

With the result found in part (c), show that $\alpha^x = 0$ and $\alpha^y = 4GM/b$.

This relation for the deflection angle captures the essence of the deflection of light by gravitational field in General Relativity. The first experimental observation was due to Eddington A. and his team in May 1919, where a deflection angle of 1.75 arc seconds was observed for stars (belong to the constellation Taurus) near the Sun during a solar eclipse.

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(adapted from Cosmology course taught in 2017 at NTU)

8 Talk in the Vacuum

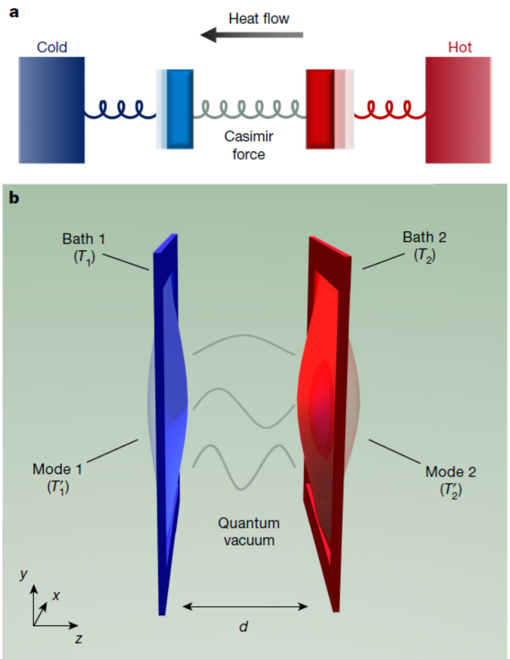

Figure 5: Casimir effect driven by quantum vacuum fluctuations between two membranes.

High school physics tells us that we will be incapable of hearing each other in the outer space due to the fact that there exists no medium in the vacuum. In this question, we deal with this problem with a more advanced perspective and see whether it is possible for us to arrive with different conclusion. As a starting point, consider two parallel uncharged conductive plates in a vacuum (see Fig. 5). Classical picture will show us nothing simply because there is no field, and thus no interactions. In quantum electrodynamics (QED) framework, however, things get different – when we put these two plates close enough, the so-called Casimir force (can be attractive or repulsive depending on the nature of the boundary) arises between them.

Now let us replace the plates with two membranes, which could be regarded as the simplification of a vocal cord and an eardrum.

(a) [1 point] Write down the dynamic equations for the vertical displacement $u_i(x, y)$ of these two membranes with built-in tensile stress as Fig. 6 shows. Assume that the two membranes have the same density ρ and thickness w, but different stress σ_1 and σ_2.

(b) [1 point] Write down the fundamental eigenmode profiles for tensile-stressed membranes.

(c) [1 point] Expand the Casimir force term to the first order, and integrate the eigenmode profile on both sides of the dynamic equations with the correction factors $\alpha_i, i \in \{1, 2\}$:

$$\alpha_i = \frac{4}{L_i^2} \left[\int_{-\min\{L_1,L_2\}/2}^{\min\{L_1,L_2\}/2} dx \cos\frac{\pi x}{L_1} \cos\frac{\pi x}{L_2} \right]^2 . \tag{1}$$

(d) [2 point] Prove that the dynamic equations can be simplified to:

$$\begin{aligned} \ddot{u}_1 + \Omega_1^2 u_1 - 2\Omega_1 g_C (u_1 - \alpha_1 u_2) &= 0 \\ \ddot{u}_2 + \Omega_2^2 u_2 - 2\Omega_2 g_C (u_2 - \alpha_2 u_1) &= 0 , \end{aligned} \tag{2}$$

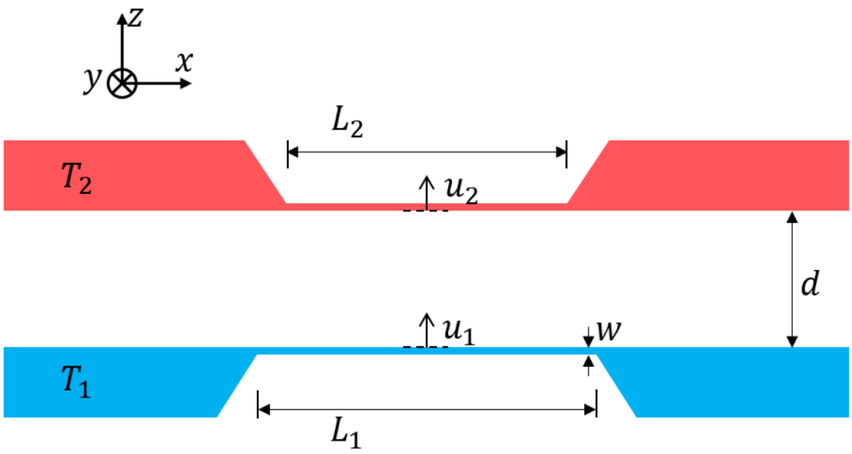

Figure 6: Schematics of the structure under theoretical consideration.

where $u_i(x, y)$ denotes vertical displacement and $\Omega_i = \pi(\sqrt{2\sigma_i/\rho})/L_i$ are the resonance frequencies and the coupling rate $g_C = F'_{\text{Cas}}(d)/2\Omega\rho w$.

(e) [1 point] When only one membrane is allowed to move, how much does the presence of Casimir force cause the membrane resonance frequency to shift? Explain the physical significance of the coupling rate g_C.

(f) [1 point] We now need to figure out a way to simulate the "talking" process. One option is to link these two membranes to a hot and a cold thermal bath separately. The dynamic equations also needs to be modified. Write down the modified dynamic equations with thermal bath.

Hint: the Langevin equation can be used to describe thermal Brownian motion of the bath:

$$m\frac{d\mathbf{v}}{dt} = -\lambda\mathbf{v} + \eta(t) \ , \tag{3}$$

where $\mathbf{v}$ is the velocity of the particle and m denotes the particle's mass. The force acting on the particle is written as a sum of a viscous force proportional to the particle's velocity. While the noise term representing the effect of the collisions with the molecules is denoted by $\eta(t)$.

(g) [1 point] By solving the equations, we can finally work out the mode temperatures of the membranes:

$$\begin{aligned} T_1' &= T_1 + \frac{\gamma_2\,(T_2 - T_1)}{(\gamma_1 + \gamma_2)\,(1 + \gamma_1\gamma_2/g_C^2)} \\ T_2' &= T_2 + \frac{\gamma_1\,(T_1 - T_2)}{(\gamma_1 + \gamma_2)\,(1 + \gamma_1\gamma_2/g_C^2)} \ , \end{aligned} \tag{4}$$

where T_i and γ_i denote the bath temperature and the mechanical dampings, respectively. Discuss the weak and strong coupling regimes when the distance between membranes is changed.

(h) [2 point] How can we know that the heat transfer is driven by Casimir force rather than electrostatic interactions or near-field thermal radiation in experiments?

Hint: think about the different power laws of these interactions.

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9 Boson-Fermion correspondence

There are two classes of particles, namely boson and fermion. In undergraduate courses, we know that boson and fermion obey different statistics − bosons follow Bose-Einstein statistics and fermions follow Fermi-Dirac statistics:

$$\begin{aligned} \text{Fermi-Dirac statistics:} \quad \langle n_i \rangle &= \frac{1}{e^{\beta(\epsilon_i - \mu)} + 1} \\ \text{Bose-Einstein statistics:} \quad \langle n_i \rangle &= \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1} \ , \end{aligned} \tag{1}$$

where $\langle n_i \rangle$ is the average number of particles in a single-particle state i, ϵ_i is the energy of the single-particle state i, μ is the total chemical potential, and $\beta = 1/k_{\mathrm{B}}T$. This difference, however, is more fundamental than it actually seems. In quantum statistics, these particles are essentially divided on the basis of the (exchange) symmetry of the system. The spin-statistics theorem then binds these kinds of combinatorial and spin symmetry to classify particles, which later becomes known as bosons and fermions. In other words, all particles that move in $3+1$D (3 spatial dimensions and 1 temporal dimension) have either integer spin or half-integer spin, and it can be shown that half-integer spin particles cannot be bosons and integer spin particles cannot be fermions. In this question, we investigate the $1+1$D case in which one could turn these two kind of quantum statistics into one another. In the context of quantum field theories, this is known as constructive bosonization which serves to solve certain interacting fermion fields.

The starting point is to write the total energy of n interacting bosons as

$$E_n = n\epsilon + Un(n-1) \ , \tag{2}$$

where $U > 0$ is a parameter describing the repulsion. In approaching this problem, we will work in the grand canonical ensemble.

(a) [1 point] Explain briefly the reason we do not choose to work in canonical ensemble.

Hint: the grand partition function can be expressed as

$$\mathcal{Z} = \sum_{N=0}^{\infty} e^{\beta\mu N} Z_N \ , \tag{3}$$

where Z_N denotes the N-particle canonical partition function that reads

$$Z_N = \sum_i e^{-\beta E_i} \ . \tag{4}$$

(b) [1 point] Show that the grand partition function can be expressed as

$$\mathcal{Z} = \prod_j \left(\sum_{n_j} \exp\left(-\beta\left(n_j(\epsilon_j + U(n_j - 1) - \mu)\right)\right) \right) \ . \tag{5}$$

(c) [2 points] Evaluate the average occupation of the state n_k knowing that the probability of the system to be in state i is given by

$$P_i = \frac{1}{\mathcal{Z}} \exp\left(-\beta\left(E_i - \mu N_i\right)\right) \ , \tag{6}$$

while the occupancy number can be obtained using the relation

$$\langle n_i \rangle = \sum_R n_i P_R \ . \tag{7}$$

Note that the state R of the many-particle system can be specified by the occupancy of the single-particle states.

Hint: at this point, it is not practical to compute the infinite summation or infinite product in $\langle n_k \rangle$.

(d) [1 point] By sending $U \to 0$, show that one could recover the Bose-Einstein distribution.

(e) [1 point] What kind of distribution does the particle follow in the regime $U \to \infty$? Briefly explain the physical relevance of sending U to infinity. This case has been studied for typical systems with very large repulsive interactions at close range, such as $\mathrm{He}-4$.

(f) [2 points] Consider one single lattice site and suppose Charlene claims that she can provide a mapping between spin-1/2 particle and the particle in part (e). Give an argument to either support or reject her claim.

(g) [2 points] Charlene stands with her claim, nonetheless. She proceed to identify the creation and annihilation operators of the particle in part (e) with the spin raising and lowering operators of the spin-1/2 particle. She denotes the creation and annihilation operators as $c^\dagger$ and c, respectively. What is the relation between $c^\dagger$ and c to the spin raising and lowering operators that she gets? Also, help her to verify the commutation relation $[c, c^\dagger] = 1 - 2n$ where $n = c^\dagger c$.

Hint: the raising and lowering operators are $S^+ = S_x + iS_y$ and $S^- = S_x - iS_y$, respectively. In the fundamental representation of $SU(2)$, the operators S_i takes the form of $S_i = \sigma_i/2$ where σ_i are the Pauli matrices obeying $\sigma_a \sigma_b = \delta_{ab}\mathbb{1} + i\varepsilon_{abc}\sigma_c$ for $a, b, c \in \{1, 2, 3\}$. Note that $\delta_{ab} = 1$ for $a = b$ and vanishes for $a \neq b$. While $\varepsilon_{ijk} = 1$ if (i, j, k) is an even permutation of $(1, 2, 3)$, but $\varepsilon_{ijk} = -1$ if it is an odd permutation. If any index is repeated, $\varepsilon_{ijk} = 0$.

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10 Coupled oscillators

Just as a single simple harmonic oscillator has been used as models for various physical systems, a system of two coupled oscillators could be used to give intuition of dynamical properties of coupled systems. Let's consider two identical masses attached with springs such that none of them are under tension or compression as shown in the figure below.

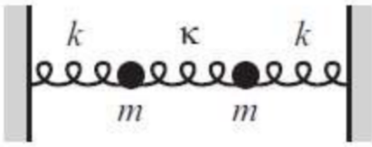

Figure 7: Two identical masses of mass m attached with springs.

Note that the spring constant of the spring coupling the two masses is $\kappa \ll k$, where k is the spring constant of springs attached to the boundaries. Let the position of the particle on the left and right be x_1 and x_2, respectively.

(a) [1 point] Prove that the equation of motion for this system is

$$\ddot{\mathbf{x}} = -\frac{1}{m}\mathbf{K}\mathbf{x} \ , \tag{1}$$

where $\mathbf{x} = (x_1, x_2)^T$, and $\mathbf{K} = \begin{pmatrix} k+\kappa & -\kappa \\ -\kappa & k+\kappa \end{pmatrix}$.

(b) [2 points] Find the normal modes of oscillation by using the ansatz $\mathbf{x} = \mathbf{C}e^{i\omega t}$, where $\mathbf{C}$ is a column vector. Given $\kappa \ll k$, what are the angular frequency of these modes?

(c) [2 points] Suppose that at $t = 0$, the particle on the left is displaced by A while the particle on the right is left at its equilibrium position. Find the evolution of x_1, x_2, and the envelopes of the oscillations. This behavior is similar to the Rabi oscillation of atoms in external field and spins in oscillating magnetic field.

(d) [1 point] Suppose that we can change the spring constants of the springs attached to the boundaries (from k to k_1 and k_2, respectively) without changing their natural lengths. Show that the equations of motion are

$$\begin{aligned} \ddot{x}_1 + \omega_1^2 x_1 &= \omega_c^2 x_2 \\ \ddot{x}_2 + \omega_2^2 x_2 &= \omega_c^2 x_1 \ , \end{aligned} \tag{2}$$

where $\omega_1^2 = (k_1 + \kappa)/m$, $\omega_2^2 = (k_2 + \kappa)/m$, and $\omega_c^2 = \kappa/m$.

(e) [1 point] Since we assume that $\kappa \ll k_1, k_2$, the masses prefer to oscillate around their natural frequencies rather than exchanging their energies. Let's move to the rotating frame of the oscillators by letting

$$\begin{aligned} x_1(t) &= \mathrm{Re}\left\{\frac{Ae^{i\omega_1 t}}{\sqrt{\omega_1}}\right\} \\ x_2(t) &= \mathrm{Re}\left\{\frac{Be^{i\omega_2 t}}{\sqrt{\omega_2}}\right\} \ , \end{aligned} \tag{3}$$

Prove that in rotating frame, and under the slow-varying amplitude assumption, the coupled equations in part (d) become

$$i\frac{\partial}{\partial t}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 & \frac{\Omega}{2}e^{i\delta t} \\ \frac{\Omega}{2}e^{-i\delta t} & 0 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} \ , \tag{4}$$

where $\delta = \omega_2 - \omega_1$ and $\Omega = \omega_c^2/\sqrt{\omega_1\omega_2}$.

(f) [1 point] Let's move to another rotating frame by letting $A = ae^{i\frac{\delta}{2}t}$ and $B = be^{-i\frac{\delta}{2}t}$, chosen by killing off the off-diagonal phases and getting

$$i\frac{\partial}{\partial t}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \delta & \Omega \\ \Omega & -\delta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}, \tag{5}$$

which is also the equation governing the evolution of a spin-1/2 spin under the Hamiltonian $H = \frac{\delta}{2}\sigma_z + \frac{\Omega}{2}\sigma_x$ (representing a static field pointing along z-direction and a static field along the x-direction).

(g) [2 points] Solve the full time evolution of x_1 and x_2. Now, suppose one starts the oscillation in the left most spring with $\omega_c \ll \omega_1 \ll \omega_2$ and slowly increases ω_1 until $\omega_c \ll \omega_2 \ll \omega_1$. Which springs are oscillating at the end? To make the thinking process easier, you can assume the springs system is now the coupled pendulum and you can change the oscillation frequency of one pendulum by changing its length.

Hint: you may find it useful denoting $\delta/\sqrt{\delta^2 + \Omega^2} = \cos\theta$.

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